Algebra: Factorising and Long Division 2

Mathematics Worksheet

This is one of a series of worksheets designed to help you increase your confidence in handling Mathematics. This worksheet contains both theory and exercises which cover:-

1. Revision of multiplying brackets
2. Factorisation
3. Long Division in Algebra

There are often different ways of doing things in Mathematics and the methods suggested in the worksheets may not be the ones you were taught. If you are successful and happy with the methods you use it may not be necessary for you to change them. If you have problems or need help in any part of the work then there are a number of ways you can get help.

For students at the University of Hull

- Ask your lecturers.
- You can contact a Mathematics Tutor from the Skills Team on the email shown below.
- Access more Maths Skills Guides and resources at the website below.
- Look at one of the many textbooks in the library.

Web:  http://libguides.hull.ac.uk/skills
Email:  skills@hull.ac.uk
Background
This Worksheet assumes that you know how to “multiply brackets out”. There are a number of different ways, some of which are given below. The final method given is used to help with factorisation and long division later in the worksheet. See also Algebra 2.

1. Multiplying brackets
Multiplying \((x + y)\) and \((a + b)\) together can be done in a number of ways
(a) The ‘eye-brows method
\[
\begin{align*}
(x + y)(a + b) &= xa + ya + xb + yb
\end{align*}
\]
Collecting terms gives
\[
(x + y)(a + b) = xa + ya + xb + yb
\]
(b) Taking \(a, b, x\) and \(y\) as lengths we have a rectangle \((x + y)\) by \((a + b)\)
\[
\begin{align*}
(a \quad x \quad y) \\
(xa \quad ya) \\
(xb \quad yb)
\end{align*}
\]
From the diagram we see that the total area is given by
\[
(x + y)(a + b) = xa + ya + xb + yb
\]
This is equivalent to adding all the bits together giving
\[
(x + y)(a + b) = xa + ya + xb + yb
\]
(c) Use a table
\[
\begin{array}{c|cc}
  x & y \\
  \hline
  a & xa & ya \\
  b & xb & yb \\
\end{array}
\]
\[
(x + y)(a + b) = xa + ya + xb + yb
\]
this method is especially useful when some of the terms are negative or the brackets contain more than 2 terms, see example (c) below, and also for factorisation and long division.

Examples Expand and simplify the following
(a) \((x-3)(2x+5)\)
\[
\begin{array}{c|cc}
  x & -3 \\
  \hline
  2x & 2x^2 & -6x \\
  5 & 5x & -15 \\
\end{array}
\]
Collecting terms gives \((x-3)(2x+5) = 2x^2 - x - 15\)
(b) 
\[(x^2 - 2xy)(x + 2y)\]

\[
\begin{array}{c|cc}
  & x^2 & -2xy \\
  x & x^3 & -2x^2y \\
 2y & 2x^2y & -4xy^2 \\
\end{array}
\]

Collecting terms gives \((x^2 - 2xy)(x + 2y) = x^3 - 4xy^2\)

(c) 
\[(2x - 4y + 3)(3x - 2y - 5)\]

\[
\begin{array}{c|ccc}
  & 2x & -4y & 3 \\
 3x & 6x^2 & -12xy & 9x \\
-2y & -4xy & 8y^2 & -6y \\
-5 & -10x & 20y & -15 \\
\end{array}
\]

Hence \((2x - 4y + 3)(3x - 2y - 5) = 6x^2 + 8y^2 - 16xy - x + 14y - 15\)

(d) 
\[(x - 2)(2x + 3) + 5\]

This could be set out as:

\[
\begin{array}{c|ccc}
  & x & -2 \\
 2x & 2x^2 & -4x & +5 \\
3 & 3x & -6 \\
\end{array}
\]

Hence \((x - 2)(2x + 3) + 5 = 2x^2 - x - 6 + 5 = 2x^2 - x - 1\)

(e) 
\[(x^2 - 3x + 5)(3x - 4) + x - 4\]

\[
\begin{array}{c|ccc}
  & x^2 & -3x & +5 \\
 3x & 3x^3 & -9x^2 & 15x \\
-4 & -4x^2 & 12x & -20 \\
\end{array}
\]

Hence \((x^2 - 3x + 5)(3x - 4) + x - 4 = 3x^3 - 13x^2 + 27x - 20 + x - 4 = 3x^3 - 13x^2 + 28x - 24\)
2 Factorisation
We can also use the table method to help us factorise quadratic expressions (see also Algebra 2):

Examples
(a) Factorise \( x^2 + 4x + 3 \)

<table>
<thead>
<tr>
<th>( A )</th>
<th>( B )</th>
</tr>
</thead>
</table>
| \( C \) | \( x^2 \) | To get the \( x^2 \) term \( A \) and \( C \) must be 1
| \( D \) | 3 | From the signs both \( B \) and \( D \) are positive;

This leads to:

\[
\begin{array}{c|c|c}
\hline
x & 1 \\
\hline
x & x^2 & x \\
3 & 3x & 3 \\
\hline
\end{array}
\]

This gives \( x \) and \( 3x \) in the other two boxes

a total \( x \)-term of \( x + 3x = 4x \) as required.

Solution Factors of \( x^2 + 4x + 3 \) are \((x + 1)(x + 3)\).

(b) Factorise \( 2x^2 - 7x + 3 \)

<table>
<thead>
<tr>
<th>( A )</th>
<th>( B )</th>
</tr>
</thead>
</table>
| \( C \) | \( 2x^2 \) | To get the \( 2x^2 \) term take \( A = 1, C = 2 \)
| \( D \) | 3 | From the signs both \( B \) and \( D \) are negative, hence
to get the \(+3\), \( B = -1, D = -3 \) or vice-versa

This leads to

EITHER

\[
\begin{array}{c|c|c}
\hline
2x & -1 \\
\hline
2x & 2x^2 & -2x \\
-3 & -3x & 3 \\
\hline
\end{array}
\]

This gives \(-2x\) and \(-3x\) in the other two boxes,
a total of \(-2x - 3x = -5x\) which is not correct

OR

\[
\begin{array}{c|c|c}
\hline
x & -3 \\
\hline
2x & 2x^2 & -6x \\
-1 & -x & 3 \\
\hline
\end{array}
\]

This gives \(-x\) and \(-6x\) in the other two boxes,
a total of \(-6x - x = -7x\) as required

Solution Factors of \( 2x^2 - 7x + 3 \) are \((x - 3)(2x - 1)\).

(c) Factorise \( 2x^2 - 5x - 3 \)

<table>
<thead>
<tr>
<th>( A )</th>
<th>( B )</th>
</tr>
</thead>
</table>
| \( C \) | \( 2x^2 \) | To get \( 2x^2 \) take \( A = 1 \) and \( C = 2 \)
| \( D \) | -3 | To get the \(-3\), \( B = 1 \) and \( D = -3 \) or vice-versa or
This gives 4 possibilities

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2x</td>
<td>2x²</td>
<td>6x</td>
</tr>
<tr>
<td>-3</td>
<td>-x</td>
<td>-3</td>
<td>3</td>
</tr>
</tbody>
</table>

(x-term total 6x – x = 5x)

From Φ factors of 2x² – 5x – 3 are (x – 3)(2x + 1)

In practice there is no need to work out Φ and Φ once you’ve found that Φ gives the correct result, except as a check.

(d) Factorise 12x² – 2x – 24

First we notice the common factor of 2 giving 12x² – 2x – 24 = 2(6x² – x – 12)

We now need to factorise 6x² – x – 12

<table>
<thead>
<tr>
<th>Ax</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cx</td>
<td>6x²</td>
</tr>
<tr>
<td>D</td>
<td>-12</td>
</tr>
</tbody>
</table>

To get the 6x² term take A = 1 & C = 6 or A = 2 & C = 3

To get the –12, B = 1 & D = -12 or B = 2 & D = -6 or B = 3 & D = -4 or vice-versa with both number and signs!

There are a lot of possibilities here but as the expression 6x² – x – 12 has no common factor then the brackets cannot have a common factor, ie we cannot have a factor 6x – 3 which has a common factor of 3 (see Algebra 2 for more details.)

Trial and error or sheer persistence gives the solution as

12x² – 2x – 24 = 2(3x + 4)(2x – 3)

Exercise 1

Factorise the following

1. x² + 3x + 2
2. 9x² – 1
3. x² – 6x + 9
4. x² – 4x + 4
5. 6x² + x – 2
6. 9x² – 6x + 1
7. x² + 2x – 8
8. x² – 6x – 16
9. 2d² + 3d + 1
10. 4x² – 16y²
11. 4a² – 16a
12. 2x² – 3x + 1
13. 6x² + 5x – 6
14. x² – 2x – 3
3. Division

We know that \( 4 \times 2 + 1 = 9 \), from this we can see that \( \frac{9}{2} = 4 \text{ rem } 1 \) or \( 4 \frac{1}{2} \).

In example (d) on page 2 we showed that \((x - 2)(2x + 3) + 5 = 2x^2 - x - 1\)

In the same way, from this, we have:

\[
\frac{2x^2 - x - 1}{x - 2} = (2x + 3) \text{ rem } 5
\]

or

\[
\frac{2x^2 - x - 1}{x - 2} = (2x + 3) + \frac{5}{x - 2}
\]

As \((x^2 + 2x - 2)(x^2 + x - 1) + 2x + 1 = x^4 + 3x^3 - x^2 - 2x + 3\)

In the same way \(\frac{x^4 + 3x^3 - x^2 - 2x + 3}{x^2 + x - 1} = (x^2 + 2x - 2) \text{ rem } 2x + 1\)

or \(\frac{x^4 + 3x^3 - x^2 - 2x + 3}{x^2 + x - 1} = (x^2 + 2x - 2) + \frac{2x + 1}{x^2 + x - 1}\)

We can use the matrix method to help with algebraic division. It seems very cumbersome when written out in the amount of detail shown. In practice you should be able to cut down on a lot of the steps.

Note, if there is a remainder

- when dividing by, say, \((x - 2)\) you can only get a number in the remainder,

- when dividing by, say, \((x^2 + 2x - 3)\) you can only get, at most, an expression of the form \(Ax + B\) in the remainder,

- In general the maximum power in the remainder will be one less than in the dividend (the term you are dividing by).

**Examples**

(a) Simplify \(\frac{2x^2 - x - 1}{x - 2}\).

Let \(\frac{2x^2 - x - 1}{x - 2} = Ax + B\) remainder \(S\) then \(2x^2 - x - 1 = (x - 2)(Ax + B) + S\)

This will come from a matrix such as:

\[
\begin{array}{ccc}
\hline
& Ax & B \\
\hline
x & 2x^2 & Px \\
-2 & Qx & R \\
\hline
\end{array}
\]

The \( x, -2 \) and \( 2x^2 \) can be put in giving \( A = 2 \) and, hence, \( Q = -4 \)

\[
\begin{array}{ccc}
\hline
& 2x & B \\
\hline
x & 2x^2 & Px \\
\hline
\end{array}
\]

The \( x \) term is \(-x\) so \(-4x + Px = -x\)

\[\Rightarrow P = 3\]
-2 | -4x  R  hence B = 3; R = -6

\[
\begin{array}{ccc}
2x & 3 \\
x & 2x^2 & 3x \\
-2 & -4x & -6
\end{array}
\]

The constant term is -1 so S - 6 = -1 \Rightarrow S = 5

so we get \( \frac{2x^2 - x - 1}{x - 2} = (2x + 3) + \frac{5}{x - 2} \).

(b) Simplify \( \frac{2x^3 - 7x^2 + 7x + 2}{2x - 1} \)

Let \( \frac{2x^3 - 7x^2 + 7x + 2}{2x - 1} = Ax^2 + Bx + C \) remainder D

then \( 2x^3 - 7x^2 + 7x + 2 = (2x - 1)(Ax^2 + Bx + C) + D \)

This comes from the following matrix

| 2x | 2x^3 | Px^2 | Qx | +D    | The 2x, -1 and 2x^2 can be put in giving A = 1 and, hence, R = -1 |
|----|------|------|----|-------|-----------------------------------------------------------------
| -1 | x^2  | Bx   | C  | +D    | x^2 term: \(-x^2 + Px^2 = -7x^2 \Rightarrow P = -6\) hence B = -3; S = 3 |
| 2x | 2x^3 | Px^2 | Qx | +D    | x term: \(3x + Qx = 7x \Rightarrow Q = 4\), hence C = 2, T = -2 |
| -1 | x^2  | -3x  | C  | +D    | constant term: \(-2 + D = 2 \Rightarrow D = 4\) |

Solution \( \frac{2x^3 - 7x^2 + 7x + 2}{2x - 1} = x^2 - 3x + 2, \text{Remainder 4} \)

or \( \frac{2x^3 - 7x^2 + 7x + 2}{2x - 1} = x^2 - 3x + 2 + \frac{4}{2x - 1} \)
(c) Simplify \( \frac{6x^5 + 7x^4 - 4x^2 + 7x + 4}{3x^2 + 2x - 1} \)

Dividing \( 6x^5 + 7x^4 - 4x^2 + 7x + 4 \) by \( 3x^2 + 2x - 1 \) will give terms starting with \( 2x^3 \) and leave a possible remainder having an \( x \)-term and a number.

Let \( \frac{6x^5 + 7x^4 - 4x^2 + 7x + 4}{3x^2 + 2x - 1} = 2x^3 + Bx^2 + Cx + D \) remainder \( Ex + F \)

Then \( 6x^5 + 7x^4 - 4x^2 + 7x + 4 = (3x^2 + 2x - 1)(2x^3 + Bx^2 + Cx + D) + Ex + F \)

<table>
<thead>
<tr>
<th>( 2x^3 )</th>
<th>( Bx^2 )</th>
<th>( Cx )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3x^2 )</td>
<td>( 6x^5 )</td>
<td>( Px^4 )</td>
<td>( Qx^3 )</td>
</tr>
</tbody>
</table>
| \( 2x \) | \( 4x^4 \) | \( Tx^3 \) | \( Ux^2 \) | \( Vx \) +F | consider \( x^4 \) term \( Px^4 + 4x^4 = 7x^4 \) \( \Rightarrow P = 3 \), hence \( B = 1 \); \( T = 2 \); \( X = -1 \)
| \( -1 \) | \( -2x^3 \) | \( Xx^2 \) | \( Yx \) | \( Z \) |

<table>
<thead>
<tr>
<th>( 2x^3 )</th>
<th>( x^2 )</th>
<th>( Cx )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3x^2 )</td>
<td>( 6x^5 )</td>
<td>( 3x^4 )</td>
<td>( Qx^3 )</td>
</tr>
</tbody>
</table>
| \( 2x \) | \( 4x^4 \) | \( 2x^3 \) | \( Ux^2 \) | \( Vx \) +F | \( x^3 \) term gives \( Qx^3 + 2x^3 - 2x^3 = 0 \) \( \Rightarrow Q = 0 \), hence \( C = 0 \); \( U = 0 \); \( Y = 0 \)
| \( -1 \) | \( -2x^3 \) | \( -x^2 \) | \( Yx \) | \( Z \) |

<table>
<thead>
<tr>
<th>( 2x^3 )</th>
<th>( x^2 )</th>
<th>( 0 )</th>
<th>( D )</th>
</tr>
</thead>
</table>
| \( 3x^2 \) | \( 6x^5 \) | \( 3x^4 \) | \( 0 \) | \( Rx^2 \) +Ex | \( x^2 \) term gives \( Rx^2 - x^2 = -4x^2 \) \( \Rightarrow R = -3 \), hence \( D = -1 \); \( V = -2 \); \( Z = 1 \)
| \( 2x \) | \( 4x^4 \) | \( 2x^3 \) | \( 0 \) | \( Vx \) +F |
| \( -1 \) | \( -2x^3 \) | \( -x^2 \) | \( 0 \) | \( Z \) |

<table>
<thead>
<tr>
<th>( 2x^3 )</th>
<th>( x^2 )</th>
<th>( 0 )</th>
<th>( -1 )</th>
</tr>
</thead>
</table>
| \( 3x^2 \) | \( 6x^5 \) | \( 3x^4 \) | \( 0 \) | \( -3x^2 \) +Ex | \( x \) term gives \( Ex - 2x = 7x \) \( \Rightarrow E = 9 \), constant term \( F + 1 = 4 \) \( \Rightarrow F = 3 \)
| \( 2x \) | \( 4x^4 \) | \( 2x^3 \) | \( 0 \) | \( -2x \) +F |
| \( -1 \) | \( -2x^3 \) | \( x^2 \) | \( 0 \) | \( 1 \) |

Solution \( \frac{6x^5 + 7x^4 - 4x^2 + 7x + 4}{3x^2 + 2x - 1} = 2x^3 + x^2 - 1 + \frac{9x + 3}{3x^2 + 2x - 1} \)

An alternative way of tackling the last example is:
let \( \frac{6x^5 + 7x^4 - 4x^2 + 7x + 4}{3x^2 + 2x - 1} = 2x^3 + Bx^2 + Cx + D + \frac{Ex + F}{3x^2 + 2x - 1} \)

Then
\[
6x^5 + 7x^4 - 4x^2 + 7x + 4 = (3x^2 + 2x - 1)(2x^3 + Bx^2 + Cx + D) + Ex + F
\]
and you can equate the coefficients of \( x \) to find the values of \( B, C, D, E \) and \( F \).

When you write the expansion this way it is not so ‘easy’ to see the coefficients but is good practice, especially for those going on to use Partial Fractions etc. The work is virtually the same as in the first method.

**Exercise 2**

Find the quotients and remainders in the following

1. \( \frac{x^2 - 3x + 7}{x - 2} \)
2. \( \frac{3x^2 + 4x - 3}{3x + 1} \)
3. \( \frac{4x^2 + 4x - 9}{2x + 5} \)
4. \( \frac{x^3 + 7x^2 + 10x}{x + 4} \)
5. \( \frac{6x^3 - 13x^2 + 16x + 6}{2x - 3} \)
6. \( \frac{10x^4 + x^3 - 7x^2 + 11x - 8}{x^2 + 1} \)
7. \( \frac{3x^4 - 9x^3 + x^2 - 8x + 13}{x - 3} \)
8. \( \frac{x^4 - x^3 + 5x + 3}{x^2 - 2x + 3} \)
9. \( \frac{6x^5 + 6x^4 - x^3 - 6x^2 - 6x - 2}{2x^2 + 2x - 1} \)

**Answers**

**Exercise 1**

1. \( (x + 2)(x + 1) \)
2. \( (3x - 1)(3x + 1) \)
3. \( (x - 3)^2 \)
4. \( (x - 2)^2 \)
5. \( (3x + 2)(2x - 1) \)
6. \( (3x - 1)^2 \)
7. \( (x + 4)(x - 2) \)
8. \( (x - 8)(x + 2) \)
9. \( (2d + 1)(d + 1) \)
10. \( 4(x - 2y)(x + 2y) \)
11. \( 4a(a - 4) \)
12. \( (2x - 1)(x - 1) \)
13. \( (3x - 2)(2x + 3) \)
14. \( (x + 1)(x - 3) \)
Exercise 2

1. \( x - 1 \) Rem 5  
2. \( x + 1 \) Rem -4  
3. \( 2x - 3 \) Rem 6  
4. \( x^2 + 3x - 2 \) Rem 8  
5. \( 3x^2 - 2x + 5 \) Rem 21  
6. \( 10x^2 + x - 17 \) Rem 10x + 9  
7. \( 3x^2 + x - 5 \) Rem -2  
8. \( x^2 + x - 1 \) Rem 6  
9. \( 3x^3 + x - 4 \) Rem 3x - 6

We would appreciate your comments on this worksheet, especially if you've found any errors, so that we can improve it for future use. Please contact the Maths tutor by email at skills@hull.ac.uk

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