This is one of a series of guides designed to help you increase your confidence in handling mathematics. This guide contains both theory and exercises which cover:-

1. Recap of vectors  
2. Unit vectors, normals and bases  
3. Vector equations of straight lines and planes  
4. Intersections of lines and planes  
5. Div, Grad, Curl & the Laplacian

There are often different ways of doing things in mathematics and the methods suggested in the guides may not be the ones you were taught. If you are successful and happy with the methods you use it may not be necessary for you to change them. If you have problems or need help in any part of the work then there are a number of ways you can get help.

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1. Recap of vectors

Before moving on to new topics, here is a quick run through of what you should already have learned from Vectors 1.

- Vectors are directed line segments - they can be represented in component form or by direction and magnitude. We can use trigonometry and Pythagoras’ theorem to switch between the forms

  - If \( \vec{a} = (x, y, z) \) then \( |\vec{a}| = \sqrt{x^2 + y^2 + z^2} \).

  - If \( \vec{a} = (x, y, z) \) and \( \vec{b} = (\alpha, \beta, \gamma) \) then \( \vec{a} \pm \vec{b} = (x \pm \alpha, y \pm \beta, z \pm \gamma) \).

  - If \( \vec{a} = (x, y, z) \) then \( K\vec{a} = (Kx, Ky, Kz) \) for any value \( K \).

- The scalar product of two vectors is used to find the angle between them and the orthogonal projection of one onto the other. The formula is: \( \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos \theta \) where \( \theta \) is the angle between the vectors and \( \vec{a} \cdot \vec{b} \) means multiply each pair of corresponding coordinates together and then add the results.

- The cross product of two vectors is used to find a vector that is perpendicular to both vectors. The formula is:

  \[
  \vec{a} \times \vec{b} = \begin{vmatrix}
  \hat{i} & \hat{j} & \hat{k} \\
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3
  \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}
  \]

  It can be checked that the vector is indeed perpendicular by performing the scalar product with the new vector and each of the original vectors.

2. Unit vectors, normals and bases

Unit Vectors

A unit vector is a vector whose length is 1. A unit vector is usually denoted by the addition of a ‘hat’ symbol, \( \hat{a} \).

It is possible to make any vector (except the zero vector, \( \vec{0} \)) into a unit vector. The formula for this transformation is:

\[
\hat{a} = \frac{\vec{a}}{|\vec{a}|}
\]

The method is to find the modulus of the vector and then to divide the vector by the modulus.

Example

\( \vec{a} = (2, 3, 5) \). \( |\vec{a}| = \sqrt{2^2 + 3^2 + 5^2} = \sqrt{38} \)

\[
\hat{a} = \frac{1}{\sqrt{38}}(2, 3, 5) \text{ or, alternatively } \hat{a} = \left\langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle.
\]

It is normally best to keep the square root sign (the surd), rather than use a decimal approximation, in order to preserve accuracy.
Exercise 1

Convert the following vectors into unit vectors:

a) \( \mathbf{a} = (1, 6, 2) \)

b) \( \mathbf{b} = (3, -2, 1) \)

c) \( \mathbf{c} = (5, 9, -3) \)

Normal vectors

A normal vector is a special type of vector. It is a vector which is perpendicular to a given line or plane.

Normal vectors are used in the process of determining the equation of a plane and are usually denoted \( \mathbf{n} \).

2D Normals

If you are working in two dimensions then you can find the normal to a line by using one vector on the line.

The simplest way to produce this normal is to swap over the \( \mathbf{i} \) and \( \mathbf{j} \) components and then change the sign on one component.

For example:
Given \( \mathbf{a} = (2, 3) \) a normal to the line containing \( \mathbf{a} \) is \( \mathbf{n} = (-3, 2) \). Equivalently, \( \mathbf{n} = (3, -2) \) is also a normal to this line.

Each line has two normals. These normals face in opposite directions. It is usually unimportant which one you use.

In order to check that you have found a normal, you can use the dot product with the vector on the line; the result should be zero.

For example:
Is \( (7, -5) \) normal to the line containing \( (5, 7) \)? \( (5, 7) \cdot (7, -5) = 35 - 35 = 0 \). The vectors are perpendicular, so \( (7, -5) \) is normal to the given line.

3D Normals

In three dimensions we look for vectors which are normal to a given plane. To do this we need to find a vector perpendicular to two vectors which lie within the plane.

The usual method for this is to use the cross product.

For example, to find the normal to the plane containing \( \mathbf{a} = (2, -1, 3) \) and \( \mathbf{b} = (-2, 4, 1) \) you would work out:
\[ \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ -2 & 4 & 1 \end{vmatrix} = ((-1) \times 1 - 3 \times 4) \hat{i} - (2 \times 1 - 3 \times (-2)) \hat{j} + (2 \times 4 - (-1) \times (-2)) \hat{k} \]
\[ = (-1 - 12) \hat{i} - (2 + 6) \hat{j} + (8 - 2) \hat{k} \]
\[ = -13 \hat{i} - 8 \hat{j} + 6 \hat{k} \]

So the normal to the plane is \( \mathbf{a} = (-13, -8, 6) \) which can be checked via the scalar product as before.

Once you have found one normal, you can find the normal that goes in the opposite direction by multiplying it by -1.

**Exercise 2**

1 Find a normal to the line containing:
   a) (6, 8)  
   b) (23, -2)  
   c) (197, 356)

2 Find a normal to the plane containing the following points:
   a) \( \mathbf{a} = (6, 3, 1), \quad \mathbf{b} = (0, 2, 5) \)  
   b) \( \mathbf{c} = (3, 1, 2), \quad \mathbf{d} = (4, 0, 3) \)  
   c) \( \mathbf{e} = (-6, 2, 4), \quad \mathbf{f} = (1, 7, 2) \)

**Bases**

A basis is a set of vectors which can be used to produce every other vector. A basis is said to 'span' a space. The vectors in a basis must be linearly independent, i.e. there is no way of adding multiples of the vectors together to produce the zero vector, other than when you multiply them all by 0.

If the vectors which form a basis are orthogonal to each other, i.e. at right angles to each other, then they are said to form an orthogonal basis.

The basis which is used most frequently is the \( \hat{i}, \hat{j}, \hat{k} \) basis, where
\[ \hat{i} = (1,0,0), \quad \hat{j} = (0,1,0), \quad \hat{k} = (0,0,1) \].

In 2 dimensions this is just \( \hat{i} = (1,0), \quad \hat{j} = (0,1) \)

(Note that the hat symbol is not always used for \( \hat{i}, \hat{j}, \hat{k} \))

Any vector in 3-dimensional space can be written as the sum of multiples of the 3 basis vectors.

**Example**
\[ (2,6,-4) = 2(1,0,0) + 6(0,1,0) - 4(0,0,1) = 2\hat{i} + 6\hat{j} - 4\hat{k} \]

There are other possible bases for 3 dimensional space, but \( \hat{i}, \hat{j}, \hat{k} \) is the simplest to use.

To express a vector in terms of another basis, you need to use simultaneous equations in order to find out the multiples required.
Example

Express the vector \((1, 3, -6)\) in terms of the basis vectors \((1, 1, 0), (1, 0, 1), (0, 1, 1)\).

Using \(\alpha, \beta, \) and \(\gamma\) to represent the unknown multiples we have:
\[(1, 3, -6) = \alpha(1, 1, 0) + \beta(1, 0, 1) + \gamma(0, 1, 1)\]

Splitting this by coordinates we get:
\[
\begin{align*}
\alpha + \beta &= 1, \\
\alpha + \gamma &= 3, \\
-\gamma &= -6.
\end{align*}
\]

Rearranging 1. gives \(\alpha = 1 - \beta\), so substituting the value of \(\alpha\) into 2 gives
\(\alpha + (1 - \beta) + \gamma = 3 \Rightarrow 2 + \gamma - \beta.\) Adding this to the last equation gives \(-4 = 2\gamma \Rightarrow \gamma = -2\). By substituting the value for \(\gamma\) back in to the other equations, we find that \(\alpha = 5\) and \(\beta = -4\).

Hence \((1, 3, -6) = 5(1, 1, 0) - 4(1, 0, 1) - 2(0, 1, 1)\)

Exercise 3

Express the vector \(a = (3, 6)\) in terms of the following bases:

a) \(\mathbf{i}, \mathbf{j}\)  

b) \((2, 1), (1, 0)\)

3. Vector equations of straight lines and planes

Vectors can be used to find the equation of a line. There are two different methods which are used depending on what information is available.

Equation of a line given direction

This requires the following information: A vector which lies on the line, and a vector which is in the direction of the line.

If \(a\) lies on the line which is in the direction of \(c\), then choose an arbitrary point \(P\) on the line, with position vector \(r\) from an origin \(O\). This produces the diagram:

To find the equation for the line we need to express \(\overrightarrow{AP}\) in terms of the other vectors. \(\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = r - a\)

However as \(\overrightarrow{AP}\) is a part of the vector \(c\), it can also be expressed as \(\overrightarrow{AP} = \lambda c\) where \(\lambda\) is a scalar.

Hence \(r - a = \lambda c\) which rearranges to \(r = a + \lambda c\), the equation of the line.

Example

Find the equation of the line through \(a = (2, 3)\) and in the direction of \(c = (1, -2)\).

Using the formula, \(r = a + \lambda c\), gives \(r = (2, 3) + \lambda(1, -2)\).

This can be simplified to \(r = (2 + \lambda, 3 - 2\lambda)\).

Hence some points on this line (found by changing the value of \(\lambda\) ) are: \((2, 3), (3, 1), (4, -1)\).
Exercise 4

1 Find the equation of the line through the point $a = (2, 6)$ and in the direction of $c = (5, 2)$

2 Find the equation of the line through the point $a = (3, 7)$ and in the direction of $c = (-1, 6)$

Equation of a line given 2 points

This requires two fixed points on the line, A and B with position vectors $a$ and $b$ respectively and P a general point on the line with position vector $r$. Here the diagram is:

Note that if $\lambda = 0$, then $r = a$ and if $\lambda = 1$, then $r = b$

Example

Find the equation of the line which passes through the points $a = (2, 5), b = (4, 9)$.

Substitute the points into the formula $r = a + \lambda(b - a)$:

$r = (2, 5) + \lambda((4, 9) - (2, 5))$ then perform the subtraction:

$r = (2, 5) + \lambda(2, 4) = (2 + 2\lambda, 5 + 4\lambda)$

Exercise 5

1 Find the equation of the line which passes through the points $a = (4, -2), b = (3, 9)$

2 Find the equation of the line which passes through the points $a = (1, 6), b = (8, -3)$

Equation of a plane

The equation of a plane requires the use of a normal vector. Given the position vector $a$ of a point on the plane and a vector $n$ which is normal to the plane, the equation of a plane is:

$$r \cdot n = a \cdot n$$

where $r = (x, y, z)$, i.e. $r$ is a general (unknown) point on the plane and the dot is the scalar product.

In order to obtain the vector equation of a plane, work out the right hand side (RHS) and leave the left alone. In order to obtain the Cartesian equation of a plane, work out both sides.
Example

Find the equation of the plane through \( a = (1, 7, 3) \) with normal \( n = (2, -8, 1) \).

First, substitute the values into the formula:
\[
\mathbf{r} \cdot (2, -8, 1) = (1, 7, 3) \cdot (2, -8, 1)
\]

To get the vector equation, perform the dot product on the RHS:
\[
(\mathbf{r} \cdot (2, -8, 1) = -51)
\]

To get the Cartesian form, also work out the LHS:
\[
(x, y, z) \cdot (2, -8, 1) = -51 \Rightarrow 2x - 8y + z = -51
\]

Exercise 6

Find the vector and Cartesian equations of the following planes:

a) Containing the point \( a = (1, 2, 5) \) and with normal \( n = (2, -4, 6) \);

b) Containing the point \( a = (3, -2, 4) \) and with normal \( n = (1, 6, 2) \)

4. Intersections of lines and planes

Intersection of two lines

Given the equations of 2 lines, \( \mathbf{r} = \mathbf{a} + \lambda \mathbf{c} \), \( \mathbf{r} = \mathbf{b} + \mu \mathbf{d} \), the lines intersect if you can find values \( \lambda \), \( \mu \) such that \( \mathbf{r}_1 = \mathbf{r}_2 \).

For example, \( \mathbf{r} = (1, 5, 4) + \lambda(4, 2, 3), \quad \mathbf{r} = (6, 7, -1) + \mu(3, 2, 11) \)

Set the equations equal to each other, so
\( (1, 5, 4) + \lambda(4, 2, 3) = (6, 7, -1) + \mu(3, 2, 11) \)

Combine the two vectors not involving \( \lambda \) or \( \mu \):
\( \lambda(4, 2, 3) = (5, 2, -5) + \mu(3, 2, 11) \).

Split into components in order to get 3 simultaneous equations:
\[
4\lambda = 5 + 3\mu, \quad 2\lambda = 2 + 2\mu, \quad 3\lambda = -5 + 11\mu
\]

Solve to find \( \lambda \) and \( \mu \).
(If you need help with this part please refer to Algebra 4.)

In this example \( \lambda = 2 \) and \( \mu = 1 \).

To find the point where the lines cross, substitute the value of \( \lambda \) or \( \mu \) into the equation for one of the lines.

Here, substituting \( \lambda \) into the first equation gives
\[
(1, 5, 4) + \lambda (4, 2, 3) = (1, 5, 4) + 2(4, 2, 3) = (1, 5, 4) + (8, 4, 6) = (9, 9, 10)
\]

You should always check this result with the other equation to ensure it is correct.
Intersection of two planes

Given the normals \((n_1\) and \(n_2)\) to the two planes, the line of intersection of the two planes can be found from:

\[
\mathbf{r} = \mathbf{c} + \lambda \mathbf{n}
\]

where \(\mathbf{n} = n_1 \times n_2\) and \(\mathbf{c}\) is a point common to both planes.

If you are given the Cartesian forms of the planes, then you can find the normal to the plane by taking the coefficients of \(x, y,\) and \(z\). For example, the normal to \(x + 3y + 4z = 24\) is \((1,3,4)\).

Example

Find the equation of the line of intersection of the planes:

\[
\mathbf{n}_1 \cdot (2,4,1) = 14, \quad \mathbf{n}_2 \cdot (1,-2,5) = 5
\]

Here we have \(\mathbf{n}_1 = (2,4,1)\) and \(\mathbf{n}_2 = (1,-2,5)\)

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
2 & 4 & 1 \\
1 & -2 & 5
\end{vmatrix} = ((4 \times 5) - (1 \times -2)) \hat{i} - ((2 \times 5) - (1 \times 1)) \hat{j}
\]

\[
= 22\hat{i} - 9\hat{j} - 8\hat{k} \quad \text{so} \quad \mathbf{n} = (22,-9,-8)
\]

Now we need a point common to both planes. The simplest way to do this is to try setting \(z = 0\). This means that we only need deal with 2 equations in 2 unknowns.

To find the common point, set \(z = 0\) in the Cartesian equations of the planes.

This gives \(2x + 4y = 14\) and \(x - 2y = 5\). Solving these simultaneous equations gives \(x = 6\), \(y = \frac{1}{2}\). Hence \(\mathbf{c} = (6,\frac{1}{2},0)\) and the equation of the line of intersection is: \(\mathbf{r} = (6,\frac{1}{2},0) + \lambda(22,-9,8)\).

Exercise 7

Find the equation of the line of intersection for the planes:

a) \(\mathbf{n}_1 \cdot (6,3,4) = 12, \quad \mathbf{n}_2 \cdot (-2,3,1) = 4\)

b) \(2x + 4y - 3z = 14, \quad x - 2y + 4z = 3\)

5. Div, Grad, Curl & the Laplacian

Div, Grad and Curl are used to give information about a vector field. (A vector field is a field where each point can be described as a vector.) All three of these require knowledge of calculus, namely differentiation and partial differentiation. If you are not familiar with these concepts or need to brush up a little, please look at the booklet Differentiation 1 before continuing.
Div, grad and curl all utilise the symbol $\nabla$ (del or nabo).

$\nabla$ represents the vector

$$
\left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}.
$$

**Divergence**

Div is used to measure the density of a vector field in order to see how quickly matter is moving into and out of a region.

Div is calculated from a vector and the result is a scalar. $\nabla \cdot F$ is used to denote divergence.

Divergence is found by taking the dot product of $\nabla$ and the vector.

\[
\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}
\]

Here $F_x, F_y, F_z$ are the components of the vector $F$.

To calculate Div differentiate the first component with respect to $x$, then differentiate the second component with respect to $y$ and differentiate the third component with respect to $z$. Finally, add the results of the 3 differentiations together.

**Examples**

1 If $F = 2x\hat{i} + 3xy\hat{j} + z^2\hat{k}$, then

$$
\nabla \cdot F = \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (3xy) + \frac{\partial}{\partial z} (z^2)
= 2 + 3x + 2z
$$

2 If $F = x^2\hat{i} + 2z^3\hat{j} - 3xy\hat{k}$ then

$$
\nabla \cdot F = \frac{\partial}{\partial x} (x^2) + \frac{\partial}{\partial y} (2z) + \frac{\partial}{\partial z} (-3xy)
= 2x + 0 + 0 = 2x
$$

Note that the results do not include $\hat{i}, \hat{j}, \hat{k}$, they are scalars.

**Exercise 8**

Find the divergence of the following vector fields:

a) $F = 4x^2y\hat{i} + 2yz\hat{j} - 3z\hat{k}$

b) $F = 3y\hat{i} + 2x^2\hat{j} + 2yz\hat{k}$

c) $F = 3\cos 2x\hat{i} + 3y\sin 2x\hat{j} + 2z\cos x\hat{k}$

d) $F = 3e^x\hat{i} + y^2e^x\hat{j} + e^{2xz}\hat{k}$
Gradient

The gradient or Grad is used to find where in the field the gradient is changing the most rapidly. The gradient function is used on a scalar and the result is a vector.

\[ \nabla f = \frac{\partial}{\partial x} (f) \hat{i} + \frac{\partial}{\partial y} (f) \hat{j} + \frac{\partial}{\partial z} (f) \hat{k} \]

Here \( f \) is the scalar that you are differentiating.

To calculate Grad differentiate the whole function with respect to \( x \). This gives the \( \hat{i} \) component. Next differentiate the original function with respect to \( y \). This gives the \( \hat{j} \) component. Finally repeat for \( z \), giving the \( \hat{k} \) component.

Examples

1 If \( f(x, y, z) = 2z^2 + 3yx \) then
\[
\nabla f = \frac{\partial}{\partial x} (2z^2 + 3yx) \hat{i} + \frac{\partial}{\partial y} (2z^2 + 3yx) \hat{j} + \frac{\partial}{\partial z} (2z^2 + 3yx) \hat{k}
\]
\[
= 3y \hat{i} + 3x \hat{j} + 4z \hat{k}
\]

2 If \( f(x, y, z) = 5xyz^2 - 2xy + \cos x \) then
\[
\nabla f = \frac{\partial}{\partial x} (5xyz^2 - 2xy + \cos x) \hat{i} + \frac{\partial}{\partial y} (5xyz^2 - 2xy + \cos x) \hat{j}
\]
\[
+ \frac{\partial}{\partial z} (5xyz^2 - 2xy + \cos x) \hat{k}
\]
\[
= (5yz^2 - 2y - \sin x) \hat{i} + (5xz^2 - 2x) \hat{j} + (10xyz) \hat{k}
\]

Exercise 9

Find the gradient vector for the following:

a) \( f = x^2 + 3yz + 2xz^2 \)

b) \( f = 3x^2 y^2 z^3 \)

c) \( f = x \cos y + \frac{1}{2} \sin x + xy \sin z \)

d) \( f = x^2 e^x + ye^{xz} \)

Curl

Curl is used to determine the amount of rotation of a vector field. If the curl of a vector field is zero, then it is said to be ‘irrotational’, i.e. it does not rotate. The curl is used on a vector and the result of calculating the curl is also a vector.

Calculating the curl is very similar to calculating the cross product of two vectors and is denoted \( \nabla \times F \).

The main difference is that the first vector that is used here is \( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \).
The formula for curl is:

\[
\nabla \times F = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_x & F_y & F_z
\end{vmatrix} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} - \left( \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \hat{j} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}
\]

To calculate the curl, first write out the determinant with \( \hat{i}, \hat{j}, \hat{k} \) as the top row, \( \nabla \) as the second row and the vector \( F \) as the third. The next step is to work it out as if it were the cross product. The difference here is that instead of finding pairs to multiply together, you are finding out which part of \( F \) will be differentiated by \( x, y, \) or \( z \).

**Examples**

1. If \( F = x^2 \hat{i} + 2z \hat{j} - 3xy \hat{k} \) then

\[
\nabla \times F = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x^2 & 2z & -3xy
\end{vmatrix} = (0 - 3y) \hat{i} - (0 - 6x) \hat{j} + (0 - 0) \hat{k} = (0 - 3x - 2) \hat{i} + (0 - 0) \hat{j} + (0 - 0) \hat{k}
\]

2. If \( F = 2xy^2z \hat{i} + 3xyz \hat{j} + z^2 \hat{k} \), then

\[
\nabla \times F = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2xy^2z & 3xyz & z^2
\end{vmatrix} = (0 - 0) \hat{i} - (0 - 2xy^2) \hat{j} + (3y - 4xyz) \hat{k} = 2xy^2 \hat{j} + (3y - 4xyz) \hat{k}
\]

**Exercise 10**

Calculate the curl of the following vector fields:

a) \( F = x^2 y \hat{i} + z^3 y \hat{j} + x \hat{k} \)

b) \( F = y^2 z \hat{i} + x^3 z \hat{j} + 3y \hat{k} \)

c) \( F = 3y \cos 2x \hat{i} + 3x \sin 2z \hat{j} + \cos y \hat{k} \)

d) \( F = z e^{xy} \hat{i} + z^2 e^x \hat{j} + xe^{3y} \hat{k} \)
Possible combinations of Div, Grad and Curl

Which combinations work depends on whether the result of the first process produces a vector or a scalar. As Grad produces a vector, it is possible to find Div(Grad) and Curl(Grad). As Div produces a scalar it is possible to find Grad(Div). As Curl produces a vector, it is possible to find Div(Grad), and as Curl also works on vectors, you can find Curl(Curl) too.

The Laplacian

The Laplacian is denoted by $\nabla^2$.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The Laplacian works on scalars and produces a scalar result.

To find $\nabla^2 f$, find the first derivative of $f$ with respect to $x$, $y$, and $z$. Then find the second derivative of each and, finally add up your results

Example

Given $f = 3x^2y + xz^2$, find $\nabla^2 f$.

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{\partial}{\partial x} (3x^2y + xz^2) + \frac{\partial}{\partial y} (3x^2y + xz^2) + \frac{\partial}{\partial z} (3x^2y + xz^2)$$

$$= \frac{\partial}{\partial x} (6xy + z^2) + \frac{\partial}{\partial y} (3x^2) + \frac{\partial}{\partial z} (2xz)$$

$$= 6y + 0 + 2x = 6y + 2x$$

Exercise 11

Find the Laplacian of the following functions:

a) $x^2 + 3yz^3$  
b) $xy^2z^4 + 3x^3y$
Answers to Exercises

Exercise 1
a) \[ \hat{a} = \frac{1}{\sqrt{41}} (1,6,2) \] b) \[ \hat{b} = \frac{1}{\sqrt{14}} (3,-2,1) \] c) \[ \hat{c} = \frac{1}{\sqrt{115}} (5,9,-3) \]

Exercise 2
1 a) (-8, 6) or (8, -6) b) (2,23) or (-2, -23) c) (-356, 197) or (356, -197)
2 a) (13, -30, 12) or (-13, 30, -12) b) (3, -1, -4) or (-3, 1, 4) c) (-24, 16, -44) or (24, -16, 44)

Exercise 3
a) \( (3,6) = 3(1,0) + 6(0,1) \) b) \( (3,6) = 6(2,1) - 9(1,0) \)

Exercise 4
Q1. \( r = (2,6) + \lambda (5,2) \) equivalently \( r = (2+5\lambda, 6+2\lambda) \)
Q2. \( r = (3,7) + \lambda (-1,6) \) equivalently \( r = (3-\lambda, 7+6\lambda) \)

Exercise 5
Q1. \( r = (4,-2) + \lambda (-11,1) \) equivalently \( r = (4-\lambda, -2+11\lambda) \)
Q2. \( r = (1,6) + \lambda (7,-9) \) equivalently \( r = (1+7\lambda, 6-9\lambda) \)

Exercise 6
a) \( \mathbf{r} \cdot (2, -4, 6) = 24 \), \( 2x - 4y + 6z = 24 \)
b) \( \mathbf{r} \cdot (1, 6, 2) = -1 \), \( x + 6y + 2z = -1 \)

Exercise 7
a) \( \mathbf{r} = (1,2,0) + \lambda (-9,-14,24) \)
b) \( \mathbf{r} = (5,1,0) + \lambda (10,-11,-8) \)

Exercise 8
a) \( \nabla \cdot \mathbf{F} = 8xy + 2z - 3 \) b) \( \nabla \cdot \mathbf{F} = 2x^2 + 2y \)
c) \( \nabla \cdot \mathbf{F} = 2\cos x - 3\sin 2x \) d) \( \nabla \cdot \mathbf{F} = e^x (3 + 2y + 2xe^{2z}) \)

Exercise 9
a) \( \nabla f = (2xy + 2z^2)\hat{i} + (x^2 + 3z)\hat{j} + (3y + 4xz)\hat{k} \)
b) \( \nabla f = (6xy^2 z^3)\hat{i} + (6x^2 yz^3)\hat{j} + (9x^2 y^2 z^2)\hat{k} \)
\( \nabla f = (\cos y + z^2 \cos x + y \sin z)\hat{i} + (-x \sin y + x \sin z)\hat{j} + (2z \sin x + xy \cos z)\hat{k} \)
d) \( \nabla f = (2x e^y + yz e^z)\hat{i} + (x^2 e^y + e^{xz})\hat{j} + (xy e^{xz})\hat{k} \)
Exercise 10

a) $\nabla \times \mathbf{F} = -3z^2 y \mathbf{\hat{i}} - y \mathbf{\hat{j}} - x^2 \mathbf{\hat{k}}$

b) $\nabla \times \mathbf{F} = (3 - 2x^3) \mathbf{\hat{i}} + (y^2) \mathbf{\hat{j}} + (3x^2 z^2 - 2yz) \mathbf{\hat{k}}$

c) $\nabla \times \mathbf{F} = -(\sin y + 6x \cos 2z) \mathbf{\hat{i}} + 3(\sin 2z - \cos 2x) \mathbf{\hat{k}}$

d) $\nabla \times \mathbf{F} = (3xe^{3y} - 2xe^x) \mathbf{\hat{i}} + (e^{3y} - e^y) \mathbf{\hat{j}} + (z^2 e^x - xe^{3y}) \mathbf{\hat{k}}$

Exercise 11

a) $\nabla^2 (x^2 + 3yz^3) = 2 + 18yz$

b) $\nabla^2 (xy^2 z^4 + 3x^3 y) = 18xy + 2xz^4 + 12xy^2 z^2$

We would appreciate your comments on this worksheet, especially if you’ve found any errors, so that we can improve it for future use. Please contact the Maths Skills Adviser by email at skills@hull.ac.uk

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