Graphs 2

Mathematics Skills Guide

This is one of a series of guides designed to help you increase your confidence in handling mathematics. This guide contains both theory and exercises which cover:-

1. Revision of straight line graphs and quadratics  
2. Cubic  
3. Quartic  
4. Higher Powers  
5. Fractional Functions  
6. Modulus Functions

There are often different ways of doing things in mathematics and the methods suggested in the guides may not be the ones you were taught. If you are successful and happy with the methods you use it may not be necessary for you to change them. If you have problems or need help in any part of the work then there are a number of ways you can get help.

For students at the University of Hull

- Ask your lecturers.
- You can contact a maths Skills Adviser from the Skills Team on the email shown below.
- Access more maths Skills Guides and resources at the website below.
- Look at one of the many textbooks in the library.

1. Revision of straight line graphs and quadratics

The graph of any equation of the form \( y = mx + c \) (or that of any equation that can be rearranged into this form) will be a straight line.

It can be shown that \( m \) is the gradient (slope of the graph) and \( c \) the intercept on the \( y \)-axis, called the gradient-intercept equation of the line.

Given a linear equation it is easy to find two points on the line (which defines the line) and then check by finding a 3rd point.

**Example** Sketch the straight lines \( y = 3x - 6 \), \( 2x - 3y = 12 \)

\( y = 3x - 6 \): when \( x = 0, y = -6 \); when \( y = 0, x = 2 \); when \( x = 3, y = 3 \)

\( 2x - 3y = 12 \): when \( x = 0, y = -4 \); when \( y = 0, x = 6 \); when \( x = 3, y = -2 \)

or in a table

\[
\begin{array}{ccc}
  x & 0 & 2 & 3 \\
  y & -6 & 0 & 3 \\
\end{array}
\]

\[
\begin{array}{ccc}
  x & 0 & 6 & 3 \\
  y & -4 & 0 & -2 \\
\end{array}
\]
Hence the sketch.

Note: $y = 3x - 6$ has gradient 3, intercept $y = -6$
$2x - 3y = 12$ can be written as $y = \frac{2}{3}x - 4$ giving gradient $\frac{2}{3}$, intercept $y = -4$

**Exercise 1**
In each of these questions sketch the graphs (a), (b),... on the same set of axes.
1. (a) $y = 3$ (b) $y = -2$ (c) $x = 2$ (d) $x = 0$
2. (a) $y = x$ (b) $y = x - 2$ (c) $y = x + 1$ (d) $y = -x$ (e) $y = -x + 2$
3. (a) $y = 3x$ (b) $y = 3x + 2$ (c) $y = -3x + 1$ (d) $y = -3x$ (e) $y = 3x - 2$
4. (a) $2y + 3x = 6$ (b) $2y + 3x = -6$ (c) $2y + 3x = 3$

**Quadratics**
The general quadratic is given by $y = ax^2 + bx + c$, where $a \neq 0$. The diagram shows the graphs of $y = 2x^2 + 12x + 19$; $y = 4x^2 - 4x + 1$ and $y = 3x^2 - 18x + 24$

The curve $y = ax^2 + bx + c$ meets the $x$-axis ($y = 0$) where $ax^2 + bx + c = 0$. The general solution of this equation is given by the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The term $b^2 - 4ac$ is called the discriminant because it discriminates between possible numbers of roots (i.e. where the graph cuts or touches the $x$-axis).

1. if $b^2 - 4ac < 0$ there are no real roots. This means that the graph does not touch the $x$-axis at all, e.g. $y = 2x^2 + 12x + 19$ above
2. if $b^2 - 4ac = 0$ there are 2 equal real roots. This means that the graph touches the $x$-axis but does not cross it, e.g. at $x = 0.5$ for $y = 4x^2 - 4x + 1 = (2x - 1)^2$ above
3. if $b^2 - 4ac > 0$ there are 2 real roots. This means that the graph crosses the $x$-axis twice, e.g. at $x = 2, x = 4$ for $y = 3x^2 - 18x + 24 = 3(x - 2)(x - 4)$ above

Note that the graphs in all 3 examples above of $y = ax^2 + bx + c$ with $a > 0$ are all $\cup$-shaped; if $a < 0$ the graphs will be $\cap$-shaped as in example (c) below.

To sketch a quadratic graph, you either need to know where the curve cuts the $x$-axis by factorising the function or write it in ‘completed square form’ (help for both of these techniques can be found in the booklet ‘Algebra 2’ available on our website: [http://libguides.hull.ac.uk/skills](http://libguides.hull.ac.uk/skills))
**Example (a)**

Sketch \( y = 2x^2 - 3x - 2 \)

\( y = 2x^2 - 3x - 2 \)

\( = (2x + 1)(x - 2) \)

when \( y = 0 \)

then \((2x + 1)(x - 2) = 0 \)

giving \( x = -\frac{1}{2} \) or \( x = 2 \)

Also \( x = 0 \Rightarrow y = -2 \) (see points on diagram).

**Example (b)**

\( y = 2x^2 + 3x + 4 \)

\( 2x^2 + 3x + 4 \) does not factorise so we complete the square.

\( y = 2x^2 + 3x + 4 \)

\( = 2(x^2 + \frac{3}{2}x + 2) \)

\( = 2\left[\left(x + \frac{3}{4}\right)^2 - \frac{9}{16} + 2\right] \)

\( = 2\left[\left(x + \frac{3}{4}\right)^2 + \frac{23}{16}\right] \)

\( = 2\left(x + \frac{3}{4}\right)^2 + \frac{23}{8} \)

minimum at \((-\frac{3}{4}, \frac{23}{8})\) also when \( x = 0 \) we get \( y = 4 \)

**Example (c)**

Graph is \( \cap \)-shaped due to \(-x^2\) term.

\( y = -x^2 + 3x - 2 \)

\( y = -x^2 + 3x - 2 \)

\( = -(x^2 - 3x + 2) \)

\( = -(x - 2)(x - 1) \)

\( y = 0 \Rightarrow x = 2, x = 1 \)

Also when \( x = 0 \), \( y = -2 \)

as shown.

Completing the square gives

\( y = -x^2 + 3x - 2 \)

\( = -(x^2 - 3x + 2) \)

\( = -\left[(x - \frac{3}{2})^2 - \frac{9}{4}\right] \)

\( = -(x - \frac{3}{2})^2 + \frac{9}{4} \Rightarrow \) maximum at \(\left(\frac{3}{2}, \frac{1}{4}\right)\)

Where possible, factorising is the easier option. Alternatively, you can also find the co-ordinates of the points where the curve cuts the \( x \)-axis (if it does) by using the formula.
**Exercise 2**

Sketch the graphs of

1. \( y = x^2 + 3x + 2 \)
2. \( y = x^2 + 4x + 1 \)
3. \( y = -x^2 + 4x - 2 \)
4. \( y = 3x^2 + 3x + 2 \)
5. \( y = 3x^2 + 4x + 1 \)
6. \( y = -2x^2 + 4x - 1 \)

For extra practice with straight line and quadratic graphs please refer to the booklet Graphs 1, available on our website: [http://libguides.hull.ac.uk/skills](http://libguides.hull.ac.uk/skills).

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**2. The Cubic**

Cubics are equations where the highest power is 3.

For a cubic graph there are many possibilities, 5 are shown above, from left to right they are:

- \( y = (x+3)(x+2)(2x+3) \) cutting the \( x \)-axis at \( x = -3, -2, -1.5 \) (3 single points)
- \( y = (2x+1)^2(2x-1) \) cutting the \( x \)-axis at \( x = -0.5 \) (twice) and \( x = 0.5 \) (1 double & 1 single point; a double point implies the curve touches the \( x \)-axis)
- \( y = (x-2)^3 \) cutting the \( x \)-axis at \( x = 2 \)
  (1 triple point; a triple point implies a point of inflexion)
- \( y = 4(x-3)(x^2 -10x + 26) \) cutting the \( x \)-axis at \( x = 3 \) (1 single point)
- \( y = (x-5)(x^2 -10x + 27) \) cutting the \( x \)-axis at \( x = 5 \) (1 single point)

There are other possibilities for positive \( x^3 \) graphs and there are the negative cubic curves as well – reflections of those above!

There is no easy way of distinguishing which situation you have. If the equation factorises then it is easy to find the points where the curve crosses the \( x \)-axis. If it doesn’t then you may have to use approximate methods to find where the curve crosses the \( x \)-axis.

It is not easy to sketch a cubic unless you can factorise the equation. There is no simple equivalent to completing the square as with the quadratic. It is possible to find some information and make a sketch by finding the turning points using calculus. It is also possible to find approximately where the curve crosses the \( x \)-axis by using numerical methods.

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**3. The Quartic**

Quartics are equations where the highest power is 4.

As is becoming fairly obvious, the higher the power, the more possibilities there are. For a quartic you can have any number of values from 0 to 4! One version of the quartic graph is shown below. The lines shown cut the curve in points with real \( x \)-values and non-real \( x \)-values:-
The above sketch is a positive \( x^4 \) curve. Similar possibilities exist for negative \( x^4 \) curves. There are a number of other possible quartics, some of which are shown here. As with cubics, sketching is not easy unless you can factorise the equation.

Some other possible quartics:

The \( x \)-axis meets curve \( h \) in 4 points (all the same);

curve \( i \) in 2 real, not equal points;

curve \( j \) in 4 points (3 are the same);

4. Higher powers
These will give even more possibilities. In practice you need to be able to factorise the equation or find turning points to make a reasonable sketch.

There are some other clues you can find. When \( x \) gets very large (positive or negative) the highest power of \( x \) dominates all the others. For instance, in the expression \( y = x^5 + 10000x^4 - x^2 + 3x - 2 \) when \( x \) gets very large, say, \( x = 10^{10} \) then the \( x^5 \) term is \( 10^{50} \) while the next term is 'only' \( 10000 \times 10^{40} = 10^{44} \) which is one millionth of \( 10^{50} \). For large (positive or negative) values of \( x \), the term with the highest power is the most important.

Also when \( x \) is very small (positive or negative) we can see that the lowest power of \( x \) is most important. For instance when \( x = 10^{-2} \) the expression above is

\[
y = 10^{-10} + 10000 \times 10^{-8} - 10^{-4} + 3 \times 10^{-2} - 2 = -1.97 \text{ (to 2 decimal places), the value of } 3x - 2 = -1.97 \text{ so the graph of the above function will be like } y = 3x - 2 \text{ close to the line } x = 0.
\]
We have 3 pieces of information:
1. When \( x \) is large and positive then \( y \) is even larger and positive.
2. When \( x \) is large and negative then \( y \) is even larger and negative.
3. When \( x \) is small the curve behaves like \( y = 3x - 2 \).

Putting these 3 clues on a diagram gives some idea of what the graph looks like but not enough to sketch it more fully.

**Example**
If you are asked to sketch the curves
\[
\begin{align*}
y &= x^6 + 100x^4 + 2x - 5 \\
y &= x^6 - 3x^4 + 2x - 5
\end{align*}
\]
all you can say is that, in both cases as \( x \to \infty \), \( y \) behaves like \( x^6 \) (ie larger and positive) and, when \( x \) is close to 0, the curve approximates to \( y = 2x - 5 \) hence lines and arrows in the sketch.

The graphs of \( y = x^6 + 100x^4 + 2x - 5 \) and \( y = x^6 - 3x^4 + 2x - 5 \) are:

Approximate values are given where the curves cross the \( x \)-axis.

To sketch the shape below the \( x \)-axis is impossible without further information! Hence it is very difficult to sketch the curve accurately unless you can simplify or factorise the function.

Finding turning points is not easy here. In the first case turning points occur where the gradient function is \( 6x^5 + 400x^3 + 2 = 0 \) and in the second case where the gradient function \( 6x^5 - 12x^3 + 2 = 0 \). Neither can be solved easily!

**5. Sketching Fractional Functions**
Without using Calculus, there are a number of clues that can be used when sketching curves, some of which have been mentioned above. They are

1. Are there any values of \( x \) that cannot be used?
2. What happens when \( x \to +\infty \) and \( x \to -\infty \)?
3. What is the value of \( y \) when \( x = 0 \)?
4. Is there any value of \( x \) that makes \( y = 0 \)?
5. When is the function positive & when is it negative?
Dealing with large values of \( x \) is very important when sketching fractional functions. We have seen that the highest powers are the most significant terms so

\[
\frac{x^2 + 10x - 999}{x + 3245} \text{ will be close to } \frac{x^2}{x} = x \text{ when } x \text{ is very large}
\]
as the \( x^2 \) term on the top and the \( x \) term on the bottom are most important.

Also for large \( x \)

\[
\frac{3x^3 + 100x - 999}{x^2 - 23245} \approx \frac{3x^3}{x^2} = 3x, \quad \frac{(x - 3)(x + 4)}{(x^2 + 1)(x - 5)} \approx \frac{x^2}{x^2} = \frac{1}{x} \approx 0
\]

\[
\frac{3x^2 + 100x}{6x^3 + 23x^2} \approx \frac{3x^2}{6x^3} = \frac{1}{2x} \approx 0, \quad \frac{4x^4 + 6x - 9}{(x^2 + 1)(x - 3)(x + 4)} \approx \frac{4x^4}{x^4} = 4
\]

**Example (a)** Sketch the curve given by \( y = \frac{1}{x - 2} \)

1. When \( x = 2 \), \( x - 2 = 0 \) giving \( y = \frac{1}{0} \) which is not defined - it is impossible to divide by 0, hence we cannot have \( x = 2 \).

   \( x = 2 \) is called an asymptote - vertical asymptotes cannot be crossed.

2. When \( x \to +\infty \), \( y \to +0 \) i.e. small and positive (if \( x = 10^6 \) then \( y \approx 10^{-6} \))

   when \( x \to -\infty \), \( y \to -0 \) i.e. small and negative (if \( x = -10^6 \) then \( y \approx -10^{-6} \))

   \( y = 0 \) is a horizontal asymptote (horizontal asymptotes may, occasionally, be crossed)

3. When \( x = 0 \) \( y = -\frac{1}{2} \)

4. \( y = 0 \Rightarrow \frac{1}{x - 2} = 0 \), not possible hence \( y = 0 \) is not crossed

5. \( \frac{1}{x - 2} > 0 \Rightarrow x - 2 > 0 \). Hence for \( x > 2 \), \( y > 0 \) and for \( x < 2 \), \( y < 0 \). (See the booklet 'Inequalities and Modulus', available on our website: [http://libguides.hull.ac.uk/skills](http://libguides.hull.ac.uk/skills) for more information.)

From 1 put in the line \( x = 2 \) as an Asymptote noting that the curve cannot cross it. (Note that asymptotes are drawn as dotted lines.)

from 2 put on the asymptote \( y = 0 \) and the arrows,

from 3 put in the dot,

from 4 the curve doesn’t cross the asymptote \( y = 0 \)

from 5 we know that the lines lie in the shaded areas.

From all these clues we get the sketch of the function

\[
y = \frac{1}{x - 2}
\]
**Example (b)** Sketch the function \( y = \frac{3x}{x + 2} \)

1. \( x + 2 \) must not be zero hence \( x = -2 \) is a vertical asymptote

2. As \( x \to \pm \infty \), \( y \to \frac{3x}{x} = 3 \); hence \( y = 3 \) is an asymptote but note also that putting \( y = 3 \) gives no solution for \( x \), hence the curve does not cross \( y = 3 \)

3. When \( x = 0 \), \( y = 0 \)

4. \( y = 0 \) when \( x = 0 \) (doesn’t really add anything!)

5. \( \frac{3x}{x + 2} > 0 \Rightarrow 3x(x + 2) > 0 \) solution, from sketch below, \( x < -2 \) and \( x > 0 \)

   note also, from the sketch, that \( 3x(x + 2) < 0 \) for \(-2 < x < 1\)

Putting all the clues onto a set of axes gives:

(Now this not the same scale as the previous diagram.)

As the lines lie in the shaded regions then the graph of \( y = \frac{3x}{x + 2} \) is

**Example (c)** Sketch the function \( y = \frac{4}{(x + 2)(x - 1)} \)

1. \((x + 2)\) and \((x - 1)\) cannot be zero hence \( x = -2, x = 1 \) are vertical asymptotes

2. As \( x \to \pm \infty \), \( y \to \frac{4}{x^2} \to +0 \); hence \( x = 0 \) is a horizontal asymptote

3. When \( x = 0 \), \( y = -2 \)

4. \( y = 0 \), no value for \( x \)

5. \( \frac{4}{(x + 2)(x - 1)} > 0 \)

   \( \Rightarrow (x + 2)(x - 1) > 0 \)

From the sketch \( y > 0 \) when \( x < -2 \) and \( x > 1 \) and hence \( y < 0 \) for \(-2 < x < 1\).
Putting these on a set of axes gives:

The graph of
\[ y = \frac{4}{(x+2)(x-1)} \]
is

**Example (d) Sketch**
\[ y = \frac{x^2(x-3)}{(x+3)(x-2)} \]

1. \((x+3)\) and \((x-2)\) cannot be zero, hence \(x = -3, x = 2\) are asymptotes

2. As \(x \to \pm \infty, y \to \frac{x^3}{x^2} = x;\) \(y = x\) is an asymptote, (which cuts the curve at \(x = 0, x = 1.5\))

3. When \(x = 0, y = 0\)

4. \(y = 0\) when \(x^2(x-3) = 0 \Rightarrow x = 3, x = 0\) (twice); (ie curve touches axis at \(x = 0\))

5. \(\frac{x^2(x-3)}{(x+3)(x-2)} > 0 \Rightarrow x^2(x-3)(x+3)(x-2) > 0\)
From a sketch of
\[ y = x^2(x - 3)(x + 3)(x - 2) \]
y > 0 for
-3 < x < 0,
0 < x < 2,
x > 3
(note when x = 0 then y = 0.)

Putting the information on a set of axes (note this is not the same scale as the previous sketch) gives:

The graph of
\[ y = \frac{x^2(x-3)}{(x+5)(x-2)} \]
(Note from 4 above the curve touches the x-axis at x = 0.

Also the asymptote y = x cuts the curve at x = 0, 1, 1.5)

Exercise 3 Sketch the graphs of the following functions
\begin{align*}
1. \quad y &= \frac{x-1}{x-2} \\
2. \quad y &= \frac{x^2-1}{x-2} \\
3. \quad y &= \frac{x}{(x-1)^2} \\
4. \quad y &= \frac{-x}{(x-1)(x-3)} \\
5. \quad y &= \frac{3-2x}{(x-2)(x+2)} \\
6. \quad y &= \frac{x^2-2x-3}{x-2} \\
7. \quad y &= \frac{x^2-1}{x^2-4} \\
8. \quad y &= \frac{x^2+1}{x^2+4} \\
9. \quad y &= \frac{x}{x^3-1} \\
10. \quad y &= \frac{x(x-2)}{x^4-1} \\
\end{align*}
(hint factorise x^3 - 1) (hint factorise x^4 - 1)

6. Modulus Functions
(For help on modulus see the booklet ‘Inequalities and Modulus’, available on our website http://libguides.hull.ac.uk/skills)

Example Sketch the graph of \[ y = \left| \frac{3x(x-3)}{(x+1)(x-2)} \right| \]
The graph of \( y = \frac{3x(x-3)}{(x+1)(x-2)} \) is:

hence the graph of \( y = \frac{3x(x-3)}{(x+1)(x-2)} \) is:

(note the sections below the \( x \)-axis in the top diagram have been reflected in the \( x \)-axis)

Exercise 4 Sketch the following

1. \( y = |2x + 5| \)  
2. \( y = |8 - 7x| \)  
3. \( y = \frac{1}{x - 3} \)  
4. \( y = \left| \frac{x}{x - 3} \right| \)
5. \( y = \left| \frac{x - 1}{x + 2} \right| \)  
6. \( y = \left| \frac{x^2 - 1}{x + 2} \right| \)  
7. \( y = \left| \frac{x^2}{x - 1} \right| \)  
8. \( y = \left| \frac{3 - 2x}{x^2 - 4} \right| \)

Answers

Exercise 1

Exercise 2
Exercise 3

asymptotes $x = 2, y = 1$

asymptotes $x = 1, y = 0$

asymptotes $x = -2, y = 2, y = 0$

asymptotes $x = -2, x = 2, y = 1$

asymptotes $x = 1, y = 0$

asymptotes $x = 3, y = 0$

asymptotes $x = 1, x = 3, y = 0$

asymptotes $x = 2, y = x$ (asymptote cuts curve at $x = \frac{1}{2}, x = 1/2$)

asymptotes $x = 2, y = x$

Exercise 4

asymptotes $x = -1, x = 1, y = 0$

asymptotes $x = -1, x = 1, y = 0$
We would appreciate your comments on this worksheet, especially if you’ve found any errors, so that we can improve it for future use. Please contact the Maths Skills Adviser by email at skills@hull.ac.uk

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