This is one of a series of guides designed to help you increase your confidence in handling mathematics. This guide contains both theory and exercises which cover:

1. Introduction
2. Straight line graphs
3. Graphs of quadratics
4. Common types of graph

There are often different ways of doing things in mathematics and the methods suggested in the guides may not be the ones you were taught. If you are successful and happy with the methods you use it may not be necessary for you to change them. If you have problems or need help in any part of the work then there are a number of ways you can get help.

For students at the University of Hull
- Ask your lecturers.
- You can contact a maths Skills Adviser from the Skills Team on the email shown below.
- Access more maths Skills Guides and resources at the website below.
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1. Introduction
Graphs provide visual information. They can be used to give an overview of a situation, such as the change in interest rates over a period of years, without the reader having to interpret tables of values.

This booklet will focus on graphs of functions, rather than bar charts, pie charts or other types of graph.

Axes
Graphs are drawn with axes. There is a horizontal axis and a vertical axis. These are used to measure in the horizontal and vertical directions. The horizontal axis measures the independent variable and the vertical measures the dependent variable.

The independent variable is the variable that you can change. The dependent variable is so called as it depends on the value that has been used for the independent variable.

For example in \( y = 3x + 2 \), \( y \) is the dependent variable as its value depends on the result of \( 3x + 2 \), which will change as \( x \) changes. So a graph of this function would have \( x \) as its horizontal axis and \( y \) as its vertical axis, as shown below.

Notes
1. Both \( x \) and \( y \) are zero where the axes cross. This point is known as the origin.
2. The end of each axis is marked with an arrow. This shows that the values of \( x \) and \( y \) on the axes increase as they travel away from the origin.
3. Each axis is labelled to show which variable it represents.
4. The values on the horizontal axis are written below the axis and the values for the vertical axis are written to the left of the axis.
5. It is usual to write the equation of the graph next to the graph itself. This is particularly important when displaying more than one graph on the same axes, in order to make it easier to tell them apart.

The previous image shows only part of the graph of the function \( y = 3x + 2 \), namely the values of \( y \) for \( x \) between 0 and 8. As this function is valid for all values of \( x \), it will also use negative values.

The axes can be drawn to include these negative values.
It is important to choose the end-points for the axes carefully, in order to ensure that the graph contains all of the necessary points and is of a sensible size.

For example, to graph \( y = 3x + 2 \) between \( x = 0 \) and \( x = 5 \), draw a table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td>17</td>
</tr>
</tbody>
</table>

From the table it can be seen that the largest value of \( y \) between these points is 17. Hence a sensible choice of axes would be for \( x \) to go from 0 to 5 and for \( y \) to go from 0 to 20.

**Coordinates**

Coordinates are used to give the position of a point on a graph. They comprise the value on the horizontal axis, followed by a comma, followed by the value on the vertical axis. These are contained in brackets.

For example, from the table when \( x \) is 3, \( y \) is 11. As coordinates this would be written \((3, 11)\).

[If working in 3 dimensions the brackets would contain 3 values.]

Coordinates involving negative values are written in the same way. For example a point that was -6 on the \( x \) axis and 3 on the \( y \) axis would be written \((-6, 3)\).

To read coordinates from a graph, drop lines from the point to each axis and then read off the values.

For example, to find the coordinates of the black dot, lines were dropped to the axes, leading to values of 2 on the \( x \) axis and 8 on the \( y \) axis.

Hence the coordinates of the black dot are \((2, 8)\).

Graphs can be split into 4 quadrants, which can be used to determine whether values are positive or negative:

This diagram shows where the values are positive and negative.

When a point lies on an axis, its value is zero for the other axis. For example a point lying on the \( y \) axis represents an \( x \) value of 0, as the \( y \) axis crosses the \( x \) axis where \( x \) is 0.
Exercise 1
Write down the coordinates of the highlighted points:

a) 

Plotting and sketching graphs
To plot a graph is to draw an accurate representation of the graph, usually by marking a series of points on the axes and joining them up.

To sketch a graph is to provide an overview of the general shape of the graph, with any important points marked.

This section will cover how to plot straight line graphs only. For information on plotting graphs which are not straight lines please refer to the booklet Graphs 2, which is available from the website http://libguides.hull.ac.uk/skills

Deciding on axes
As stated before, points worked out from the equation should be used to determine the largest points and thus decide the length of the axes. However, once this is done, the scale of the axes needs to be considered.

There are two points to consider here:

1. Overall size of the graph.
   It is important to ensure that the graph fits onto a single piece of paper, but is not too small to read. Check the number of available squares in each direction before starting to draw the axes.

2. The values to be used.
   The values to be plotted will determine the scale. If all of the values are whole numbers, then you can label the axes in increments of 1. If dealing with decimals then each whole number may be spread over a number of squares.

Depending on the size of the values that require plotting, it may make more sense to have a different scale for each axis. For example the equation \( y = 5x \) produces values as in the table below:

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>20</td>
</tr>
</tbody>
</table>

Clearly the \( y \) values are increasing faster than the \( x \) values. In this case a sensible scale would be to have each line on the \( x \) axis to be an increase of 1, whereas on the \( y \) axis, the increments would be in 5s.
This would lead to the following graph:

![Graph](image)

**Plotting points**
To plot a point, start with the value corresponding to the horizontal axis. Find this value on the axis and then follow it up or down until it is level with the correct place on the vertical axis. On larger graphs it may be helpful to use two rulers to help with positioning the points; one parallel to the horizontal axis to find the vertical component and the other parallel to the vertical axis to find the horizontal component.

![Rulers](image)

Here the point (0.6, 0.4) is being plotted with the aid of two rulers.

To draw a straight line graph 3 points need to be plotted. While a straight line can be drawn from 2 points, including an extra point acts as a check. If three points are plotted and these do not fall on a straight line, then an error has been made.

Once the points have been plotted, join them up with a ruler. It is best to do this initially in pencil in case of errors.

It is only possible to draw straight lines in this way because they increase/decrease at a uniform rate. Curved graphs, therefore, cannot be drawn like this.

**Exercise 2**
Sketch the graphs of:

a) \( y = 3x + 2 \) from the points (0, 2), (1, 5) and (2, 8)
b) \( y = -2x + 4 \) from the points (0, 4), (1, 2) and (2, 0)

**2. Straight line graphs**
Straight line graphs are produced from equations of the form:

\[
y = mx + c
\]

The important fact here is that the power of \( x \) can only be 1.

One way to remember this form is to think of it as:
‘\( y \) equals \( m \)ultiple of \( x \) plus a \( c \)onstant’

The values of \( m \) and \( c \) may be whole numbers, decimals, fractions or irrational numbers.
For example, \( y = 3x + 2 \), \( y = \frac{1}{2}x - \frac{3}{19} \) and \( y = \pi x - 0.567 \) all have straight line graphs.

Sometimes it may be necessary to rearrange an equation in order to find out if it has a straight line graph.

For example, \( \frac{y + 7}{6} = x \) can be rearranged to \( y = 6x - 7 \), and 
\[ 3y - 9x = 2 \quad \text{becomes} \quad y = 3x + \frac{2}{3}. \]

Having equations written in the form \( y = mx + c \) makes them easier to draw. This is because the values \( m \) and \( c \) have special significance.

**The meaning of \( m \)**

\( m \) represents the gradient of the graph. The gradient provides information on the steepness of the slope of the graph.

- Positive gradient: slopes upwards
- Negative gradient: slopes downwards
- Zero gradient: graph is flat

The larger the gradient is (positive or negative), the steeper the slope.

In essence the gradient shows how far up (or down) the graph goes each time 1 is added onto the horizontal value.

For a straight line graph the gradient is a constant value, so this information can be used to help draw the graph.

**The meaning of \( c \)**

\( c \) is a constant value and so is not affected when the values of variables are changed. For example in \( y = 3x + 2 \), the 2 will be the same regardless of the value of \( x \) or \( y \).

When \( x = 0 \), any term involving \( x \) will also = 0. In the most general case, \( y = mx + c \), this means that the right hand side of the equation is reduced to just \( c \). Hence \( y = c \) when \( x = 0 \).

This value is known as the \( y \)-intercept. It is where the graph crosses the \( y \) axis.
The images below show the effect that the value of $c$ has on a graph.

$c = 0$  $c > 0$  $c < 0$

**Exercise 3**
For the following equations, write down the values of the gradient and the $y$-intercept:

a) $y = 6x - 8$  
b) $y = -5x + 13$  
c) $y = \frac{1}{2}x + \frac{1}{3}$

**Finding the equation of a straight line from the graph**
Sometimes it may be necessary to derive an equation by looking at the graph it produces.

To find the value of $c$, read off the value of $y$ when $x = 0$, i.e. when the graph crosses the $y$ axis.

To find the value of $m$, the gradient needs to be calculated.

The gradient is found by dividing the difference in vertical coordinates by the difference in horizontal coordinates between two points. This is sometimes called 'rise over run'.

For example, if a graph contains the points $(3, 1)$ and $(6, 10)$, then the 'rise' is $10 - 1 = 9$ and the 'run' is $6 - 3 = 3$. This makes the gradient $\frac{9}{3} = 3$.

(Note that it is always the point with the lowest $x$ value that is subtracted from the other point. The run is always positive.)

**Exercise 4**
Find the equations of each of the following graphs by reading off the intercept and working out the gradient:

a) 

b)
Finding the equation of a straight line from two points

If it is known that the graph is of a straight line, then it must take the form \( y = mx + c \).
Both of the points must fit this rule.

First find the value of the gradient. Using the example from the previous section, with points (3, 1) and (6, 10), the gradient was found to be 3.

Substitute the value of the gradient into the general formula to give \( y = 3x + c \). As the points fit this rule, substitute in the values for \( x \) and \( y \) for one of the points. In this case (3, 1) is used, giving:
\[
1 = (3 \times 3) + c \Rightarrow 1 = 9 + c
\]
Solving this gives \( c = -8 \), hence the points (3, 1) and (6, 10) belong to the graph of \( y = 3x - 8 \).

Note that it is always a good idea to check this result by substituting in the other point.

**Exercise 5**
Find the equations of the graphs containing the points:
a) (1, 9) and (2, 15)  b) (1, 3) and (3, -1)

3. Graphs of quadratics

Quadratics are functions of the form \( y = ax^2 + bx + c \), where \( a \neq 0 \).
Some examples of quadratics are:
\[
y = x^2, \quad y = 3x^2 - 4x + 3, \quad y = \frac{1}{2}x^2 + \frac{2}{5}.
\]

Graphs of quadratics are not straight line graphs, they are curves. However they do have quite a uniform shape, which makes it easier to sketch them.

When \( a \) is positive the graph is u-shaped
When \( a \) is negative the graph is n-shaped

The value of \( a \) determines how steep the graphs are. For example, compare \( y = x^2 \) with \( y = 2x^2 \) and \( y = \frac{1}{3}x^2 \).

The value of \( c \) acts in the same way as for straight line graphs; it is the intercept on the \( y \) axis.

**Roots of a quadratic**

As well as the overall shape, the location of any roots is needed in order to sketch a quadratic.

Roots are where the graph crosses or touches the horizontal axis.
With quadratics there are three distinct possibilities, either:
1. The graph does not touch the horizontal axis at all, in which case it is said to have no real roots. (The roots of these graphs are complex.)
2. The graph has two distinct roots. This means that the graph crosses the axis, changes direction (at its max/min) and then crosses the axis again.
3. The graph touches the axis but does not cross it. In this case it is considered to have one repeated root.

There are two main methods for finding the roots of a quadratic, factorising and using the quadratic formula.

**Finding roots from factorised form**
(For information on how to factorise a quadratic please refer to the booklet Algebra 2, available from our website at [http://libguides.hull.ac.uk/skills](http://libguides.hull.ac.uk/skills))

Factorising the equation does not give the roots directly. Once the equation is factorised it needs to be set to 0 as the equation equals 0 at the roots. For example
\[y = x^2 + 5x + 4 = (x + 4)(x + 1) = 0\].

In order for two things to multiply to make 0, either one of them or both of them must be equal to 0. In this example, this means that either \(x + 4 = 0\) or \(x + 1 = 0\) or both. Solving these two gives \(x = -4\) and \(x = -1\). So the roots of \(y = x^2 + 5x + 4\) are \(x = -4\) and \(x = -1\), meaning that the points (-4, 0) and (-1,0) can be plotted on the graph.

In general to find the roots simply change the sign of the numbers in the brackets.

**Finding roots using the quadratic formula**
(For information on how to use the quadratic formula please refer to the booklet Algebra 4, available from our website at [http://libguides.hull.ac.uk/skills](http://libguides.hull.ac.uk/skills))

Factorising only works when the roots of the equation are whole numbers. If they are not then the method to use is the quadratic formula:

\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

Using the quadratic formula produces the roots directly.

**Exercise 6**
1. What are the roots of the following equations?
   a) \(y = x^2 + 8x + 12 = (x + 2)(x + 6)\)  
   b) \(y = x^2 + x - 20 = (x + 5)(x - 4)\)
2. Using the quadratic formula find the roots of the following equations:
   
a) \( y = x^2 + 4x - 5 \)  
b) \( y = x^2 + 2x - 7 \)

**Sketching a quadratic**

Because quadratics are curves, it would be very difficult to draw them accurately, so sketches normally suffice.

To sketch a quadratic requires the following information:
1. General shape;
2. The location of the \( y \) intercept;
3. The location of any roots or where the graph is centred

The general shape can be determined by the coefficient of the squared term. If this is positive, then this is a u-shaped graph, if it is negative, then this will be an n-shaped graph. The size of the coefficient is also important as the larger the value (positive or negative), the steeper the graph will be.

The \( y \) intercept is found as before as the value of \( c \).

If the graph has real roots these can be found by factorising or using the quadratic formula. If not then the position of the graph can be located by completing the square.

For example
   
   \[ y = 2x^2 - 2x - 4 \]

1. The coefficient of \( x^2 \) is positive so this will be a u shaped graph. It will be steeper than \( y = x^2 \).

2. When \( x = 0 \) the value of \( y \) is -4, so the \( y \) intercept is -4.

3. By factorising or by using the quadratic formula with \( a = 2 \), \( b = -2 \) and \( c = -4 \) the roots are found to be \( x = -1 \) and \( x = 2 \). These can be checked by substitution.

Using this information the graph can be sketched:

**Exercise 7**

Sketch: a) \( y = x^2 - 2x - 3 \)  
b) \( y = x^2 + 9x + 14 \)
4. Common types of graph

Just as quadratics are u or n shaped, many other types of graph have a distinct shape. This information can be helpful when sketching them. This section looks at some common graphs and note their distinguishing features.

Exponential graphs

Exponential graphs are normally used to show patterns of growth that increase as time passes. For example it may be used to model the number of bacteria over time where the bacteria multiply by splitting into 2. This would be represented by the graph \( y = 2^x \) if the colony started out as a single bacterium.

The first few points on this graph are:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
</tr>
</tbody>
</table>

It can be seen that as \( x \) increases by 1 each time, the gaps between the \( y \) values get larger and larger. This means that the graph is getting steeper and steeper.

Other examples of exponential graphs are \( y = 3^x \), \( y = 12^2x \) and \( y = e^x \).

\( y = e^x \) is a special kind of exponential graph. The value \( e \approx 2.718 \) is chosen in order to give this graph the property that at every point, the \( y \) value is equal to its gradient.

\( y = e^x \) is also the inverse function of \( y = \ln x \).

Notice how quickly the \( y \) values increase as the \( x \) values increase. For negative \( x \), the graph gets closer and closer to 0, but never actually reaches 0. 0 is called a ‘horizontal asymptote’ and may be drawn as a dotted line.

Reciprocal graphs

Taking a reciprocal of a number means dividing 1 by that number. For example the reciprocal of 4 is \( \frac{1}{4} = 0.25 \). Reciprocal graphs take a similar format, for example \( y = \frac{1}{x} \) and \( y = \frac{3}{x+4} = 3\left(\frac{1}{x+4}\right) \) produce reciprocal graphs.

As the independent variable is situated on the bottom of a fraction this can produce problems. For example, in the equation \( y = \frac{1}{x-3} \), when \( x = 3 \) the equation becomes \( y = \frac{1}{0} \), a calculation that would bring up an error message on your calculator.

Because of this it is said that these graphs ‘cannot take’ certain values of \( x \) and these values then produce ‘vertical asymptotes’. For \( y = \frac{1}{x} \), as pictured below, the point \( x = 0 \) is a vertical asymptote.
Notice how the graph shoots upwards almost vertically as it approaches 0 from the positive side, and shoots off in the opposite direction when approaching 0 from the negative side.

\[ \frac{1}{x} \]

\[ 0 \leq x \leq 10 \]

\[ -5 \leq y \leq 5 \]

\[ x = 0 \] is an asymptote, normally denoted by a dotted line. Here the asymptote coincides with the axis.

This graph also has a horizontal asymptote. As \( x \) gets larger (positive or negative) the value of \( y \) gets closer to 0. However it is not possible for \( \frac{1}{x} = 0 \) (as this suggests that \( 1 = 0 \)), so \( y = 0 \) is a horizontal asymptote.

**Square root graphs**

Finding the square root is the inverse function of squaring.

When a number is squared, it doesn’t matter if it is positive or negative, the result will be positive. However finding the square root only produces positive numbers. Hence, although \( 3^2 = 9 \) and \( (-3)^2 = 9 \), \( \sqrt{9} = 3 \). This means that the graph of \( y = \sqrt{x} \) will not produce any negative \( y \) values.

\[ y = \sqrt{x} \]

Note that the graph looks like half of a \( y = x^2 \) graph laid on its side. There are no values for negative \( x \), as \( y = \sqrt{x} \) has no real solutions for negative numbers. The rate of change of this graph decreases as the value of \( x \) increases, i.e it flattens out.

(For further information on graphs please refer to the booklet Graphs 2, available from our website at [http://libguides.hull.ac.uk/skills](http://libguides.hull.ac.uk/skills))

**Answers to exercises**

**Exercise 1**

a) (1, 2) and (3, 6)  

b) (2, -6) and (3, -9)

**Exercise 2**

a) 

b)
Exercise 3
a) Gradient=6, intercept=-8    b) Gradient=-5, intercept=13

Exercise 4
a) Using the points (2, 4) and (4, 6): gradient=$\frac{6-4}{4-2}=\frac{2}{2}=1$, intercept=2
b) Using the points (2, 2) and (4, 8): gradient=$\frac{8-2}{4-2}=\frac{6}{2}=3$, intercept=-4

Exercise 5
a) gradient=$\frac{15-9}{2-1}=\frac{6}{1}=6$, so equation becomes $y=6x+c$. Substitute in
(1, 9) to get $9=6+c \Rightarrow c=3$. Hence equation is $y=6x+3$.
b) gradient=$\frac{-1-3}{3-1}=\frac{-4}{2}=-2$, so equation becomes $y=-2x+c$. Substitute in (1, 3) to get
$3=-2+c \Rightarrow c=5$. Hence equation is $y=-2x+5$.

Exercise 6
1. a) Roots are $x=-2$ and $x=-6$    b) Roots are $x=-5$ and $x=4$
2. a) $x=1$ and $x=-5$ (note: as these are whole numbers the equation could have been factorised)
b) $x=-1+\sqrt{8} \approx 1.83$ and $x=-1-\sqrt{8} \approx -3.83$

Exercise 7
a)

\[
y = x^2 - 2x - 3
\]

b)

\[
y = x^2 + 3x - 4
\]

We would appreciate your comments on this worksheet, especially if you’ve found any errors, so that we can improve it for future use. Please contact the Maths Skills Adviser by email at skills@hull.ac.uk

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