Algebra 3

Mathematics Skills Guide

This is one of a series of guides designed to help you increase your confidence in handling mathematics. This guide contains both theory and exercises which cover:

1. Solving simple equations
2. Evaluation and transposition of formulae

There are often different ways of doing things in mathematics and the methods suggested in the worksheets may not be the ones you were taught. If you are successful and happy with the methods you use it may not be necessary for you to change them. If you have problems or need help in any part of the work then there are a number of ways you can get help.

For students at the University of Hull

- Ask your lecturers.
- You can contact a Maths Skills Adviser from the Skills Team on the email shown below.
- Access more Maths Skills Guides and resources at the website below.
- Look at one of the many textbooks in the library.
1. Solving simple equations

An equation is a mathematical way of writing a statement about the equality of two quantities. It always uses the ‘=’ sign (read ‘is equal to’ or ‘equals’) to separate the two equal quantities. For example, the statement ‘in four years time, I will be thirty-two years old’ can be written as \( A + 4 = 32 \), where \( A \) is my current age in years.

To **solve** an equation means to find the value (or the set of all of the values, if there is more than one) of the unknown quantity, which fits (or satisfies) the equation, or find the number (or set of numbers) with make the statement true. In this example there is only one solution, namely \( A = 28 \) (i.e. my present age is 28 years).

A basic principle in solving all equations is to start with the given equation which is true and to keep it true by doing the same thing to both sides. In this respect you can think of an equation as being a balance where you have to do the same thing in each balance pan to keep it level. You repeat this action as often as necessary until the unknown quantity stands on its own. In our example, starting from \( A + 4 = 32 \)

\[
\begin{align*}
A + 4 & \quad 32 \\
-4 & -4 \quad -4 \\
A & \quad 28 
\end{align*}
\]

Hence \( A = 28 \)

Different equations use different letters for the unknown (occasionally associated with the meaning of the unknown - like \( A \) for ‘age in years’) but ‘doing the same thing to both sides’ is essential for correct solutions. Many people find the idea of the ‘balance’ helpful and use it until their confidence grows.

In solving **any** equation, no matter how difficult, you can always check that you have a solution by **substituting back into the original equation**. That is, take the solution and substitute it for the unknown in each side of the original equation, do the arithmetic, and check that both sides represent the same number. In our example, we can easily check that 28 is the solution. The left-hand side of the original equation is \( A + 4 \), and when we substitute \( A = 28 \), this gives 28 + 4, which is 32 and this equals the value on the right hand side of the original equation.

**Examples**

(a) Solve the equation \( 4x - 5 = 37 \)

\[
\begin{align*}
4x - 5 & \quad 37 \\
+5 & +5 \\
4x & \quad 42 \\
\div4 & \div4 \\
x & = 10 \frac{1}{2}
\end{align*}
\]

Four times what we want is 42 divide both sides by 4 [inverse of \( \times 4 \)]
Check by substitution
Left hand side (LHS)
\[ 4x - 5 = 4 \times 10 \frac{1}{2} - 5 = 42 - 5 = 37 \]
= Right hand side (RHS)

(b) Solve the equation \( \frac{x}{3} = 5 \). (This could be written as \( \frac{1}{3} x = 5 \).)

\( \frac{x}{3} = 5 \)

Multiply both sides by 3, the inverse of \( \frac{1}{3} \):
\[ \frac{x}{3} \times 3 = 5 \times 3 \Rightarrow x = 15 \]

Check LHS \( = \frac{x}{3} = \frac{15}{3} = 5 = \text{RHS} \)

Note – in multiplication the inverse of, say, 5 is \( \frac{1}{5} \) as \( 5 \times \frac{1}{5} = \frac{5}{1} \times \frac{1}{5} = 1 \). This also shows that the inverse of \( \frac{1}{5} \) is 5 as \( \frac{1}{5} \times 5 = \frac{1}{5} \times \frac{5}{1} = 1 \).

(c) Solve the equation \( 3(5 - 2x) = 4(x + 9) - 1 \)

Unknowns appear on both sides of the equation so first we need to tidy up the equation by multiplying the brackets out and collecting terms.

Multiply out brackets
\[ 3(5 - 2x) = 4(x + 9) - 1 \]
\[ 15 - 6x = 4x + 36 - 1 \]
\[ 15 - 6x = 4x + 35 \]

Subtract 15 from both sides
\[ \text{[inverse of +15]} \]
\[ 15 - 6x - 15 = 4x + 35 - 15 \]
\[ -6x = 4x + 20 \]

Subtract 4x from both sides
\[ \text{[inverse of +4x]} \]
\[ -6x - 4x = 4x + 20 - 4x \]
\[ -10x = 20 \]

Divide both sides by \(-10 \text{ [inverse of } \times -10 \text{ ]}\)
\[ \frac{-10x}{-10} = \frac{20}{-10} \]
\[ x = -2 \]

Check in the original equation:
LHS = \( 3(5 - 2x) = 3(5 - 2 \times -2) = 3(5 + 4) = 27 \)
RHS = \( 4(x + 9) - 1 = 4(-2 + 9) - 1 = 28 - 1 = 27 \)

LHS = RHS

Note we could have added \( 6x \) to both sides at line 3 and then subtracted 35 from both sides to give \( -20 = 10x \Rightarrow -2 = x \Rightarrow x = -2 \)

(d) Solve the equation \( 3c + 6 = \frac{1 - 2c}{2} \)

Multiply both sides by 2 \( \text{[inverse of } \div 2 \text{]} \)
\[ 2(3c + 6) = 2 \left( \frac{1 - 2c}{2} \right) \]
\[ 6c + 12 = 1 - 2c \]
Add $2c$ to both sides \([\text{inverse of} -2c]\) \[8c + 12 = 1\]

Subtract 12 from both sides \([\text{inverse of} +12]\) \[8c = -11\]

Divide both sides by 8 \([\text{inverse of} \times 8]\) \[c = -\frac{11}{8}\]

It may not always be necessary but it is important to be able to check your answer:

Checking (a calculator with fractions could be useful for this)

\[
\text{LHS} = 3c + 6 = 3 \times \left(-\frac{11}{8}\right) + 6 = -\frac{33}{8} + 6 = 1\frac{7}{8}
\]

\[
\text{RHS} = \frac{1 - 2c}{2} = \frac{1 - 2 \times \left(-\frac{11}{8}\right)}{2} = \frac{1 + \frac{22}{4}}{2} = \frac{3}{2} = 1\frac{7}{8}
\]

LHS = RHS so our solution is correct.

(e) Solve the equation \[\frac{10}{x} = 21 + \frac{4}{x}\]

Multiply both sides by \(x\) \[10 \times x = \left(21 + \frac{4}{x}\right) \times x\]

\[10 = 21x + 4\]

Subtract 4 from both sides \[6 = 21x\]

Divide both sides by 21 \[x = \frac{2}{7}\]

This simplifies to

How you write your final answer depends on the accuracy you require. The exact answer is \(x = \frac{2}{7}\) but, depending on the context, an answer of \(x = 0.2857\) (to 4 decimal places) or \(x = 0.29\) (to 2 decimal places) may be appropriate. If you want to check your answer fully you must use the exact value.

For example using \(x\) as 0.29, and giving the answer to 3 d.p. gives

\[\text{LHS} \approx \frac{10}{0.29} \approx 34.483, \text{ RHS} = 21 + \frac{4}{0.29} \approx 34.793\]

To check properly

\[\text{LHS} = \frac{10}{x} = 10 + x = 10 + \frac{2}{7} = 10 \times \frac{7}{2} = 35\]

\[\text{RHS} = 21 + \frac{4}{x} = 21 + 4 \times \frac{2}{7} = 21 + 4 \times \frac{7}{2} = 21 + 14 = 35\]

\[\text{LHS} = \text{RHS (exactly)}.\]
(f) Solve the equation \[3p = \frac{7p - 10}{13} - \frac{p}{7} + 4\]

This looks complicated but if you follow the rules you can do it, though you'll probably never meet anything as bad! With practice some steps may be combined.

Multiply through by 13
\[13 \times 3p = \frac{13}{1} \times \left(\frac{7p - 10}{13}\right) - \frac{13}{1} \times \frac{p}{7} + (13 \times 4)\]
\[39p = (7p - 10) - \frac{13p}{7} + 52\]

Multiply through by 7
\[7 \times 39p = 7(7p - 10) - \frac{7 \times 13p}{7} + 7 \times 52\]
\[273p = 49p - 70 - 13p + 364\]
\[273p = 36p + 294\]

subtract 36p from both sides
\[237p = 294\]

or \[p = 1.241\] (to 4 sig figs)

Checking is not easy!! Using the decimal value we have

LHS = \[3p \approx 3 \times 1.241 = 3.723\] (to 4 sig. figs.)

\[\text{RHS} = \frac{7p - 10}{13} - \frac{p}{7} + 4 \approx \frac{7 \times 1.241 - 10}{13} - \frac{1.241}{7} + 4\]
\[\approx \frac{8.687 - 10}{13} - \frac{1.241}{7} + 4 \approx -0.101 - 0.177 + 4 \approx 3.722\] (to 4 sig. figs.)

Hence LHS \(\approx\) RHS, the solution is correct – well to 3 significant figures!

With practice you could multiply through by 13 and 7 in one go but you do need to be very careful.

**Exercise 1**

Solve the following equations:

1. \[5c - 9 = 11\]
2. \[7 = 2n + 5\]
3. \[7s = 3s + 8\]
4. \[8 + 4f - 7 = 4 - f\]
5. \[2d + 5 - 4d = 17 + d\]
6. \[2(A + 1) = 9\]
7. \[\frac{d}{2} + \frac{1}{4} = \frac{3d}{8} - \frac{1}{4}\]
8. \[3(y + 1) = 2(2y - 1)\]
9. \[\frac{x}{3} = 4\]
10. \[\frac{b + 3}{2} = \frac{b - 3}{3}\]
11. \[\frac{b + 3}{2} = \frac{b - 3}{3} + 1\]
12. \[7(n - 1) = 3(2n + 5)\]
13. \[7s = 3(s + 8) - 3\] 
14. \[\frac{1}{2}(c + 1) = 3(c - 3) + 7\]

15. \[(2d + 5)(1 - 4d) = 17 - 8d^2\] 
16. \[\frac{5x}{3} = \frac{3}{5} + \frac{3x}{2}\]

17. \[\frac{3}{4}(A + 1) = \frac{3}{4}(1 - A)\] 
18. \[\frac{1}{x} + 3 = \frac{3}{x}\]

19. \[\frac{2}{x} = \frac{3}{x - 1}\] 
20. \[3p + 2 \left(2 - \frac{5p}{9}\right) = \frac{p}{3} + 11\]

2. Evaluation and Transposition of Formulae

A formula is an equation that describes a relationship between a number of different quantities - for example, the volume \(V\) in cm\(^3\) (cubic centimetres) of a rectangular box, with length \(l\) cm, width \(w\) cm and height \(h\) cm is given by the formula \(V = lwh\).

As shown in the leaflet Algebra 1, you can work out the value of \(V\), given values for \(l\), \(w\) and \(h\), by substituting those numbers into the formula and calculating:

e.g. if \(l = 5\), \(w = 8\), \(h = 10\), then

\[V = lwh = 5 \times 8 \times 10 = 400.\]

i.e. the volume of a box 5cm by 8cm by 10cm is 400cm\(^3\).

In the formula above, \(V\) is the subject - that is \(V\) (on its own) is expressed in terms of the other quantities. This makes finding the value of \(V\), given the other values, fairly easy. If you need to find the value of, say, the height, then you may want to find the formula that gives the height in terms of the volume, width and length - that is you need to make \(h\) the subject of the formula. The rules for rearranging (or transposing) a formula are the same as those for solving an equation - ‘do the same things to both sides’.

For example, using the formula

\[V = lwh\]

\[\frac{V}{l} = \frac{lwh}{l} = wh\]

\[\frac{V}{wh} = \frac{wh}{w} = h\]

\[h = \frac{V}{lw}\]

This final line has \(h\) as the subject of the formula.

Examples

(a) The surface area of a cylinder of height \(h\) and base radius \(r\) is given by the formula \(A = 2\pi r(r + h)\). If a cylinder has surface area 282.6 cm\(^2\) and the radius of the base is 3 cm find the height of the cylinder (take \(\pi\) as 3.142).

Substitute the values in simplify the right hand side

\[282.78 = 2 \times 3.142 \times 3 \times (3 + h)\]

\[= 18.852 \times (3 + h)\]

Divide by 18.852 [inverse of \(\times 18.852\)]

\[\frac{282.78}{18.852} = 3 + h\]

\[15 = 3 + h \Rightarrow h = 12\]

Height of cylinder 12 cm.
(b) If \( V = \frac{2R}{R - r} \) find the value of \( R \) given \( V = 7, \ r = 3 \).

Substituting the values in \( 7 = \frac{2R}{R - 3} \)

Multiply both sides by \( (R - 3) \) [inverse of \( \div (R - 3) \)]

Multiply out bracket \( 7(R - 3) = 2R \)

Add 21 to both sides [inverse of \(-21\)] \( 7R - 21 = 2R + 21 \)

Subtract \( 2R \) from both sides [inverse of \( 2R \)] \( 7R - 2R = 21 \)

Dividing by 5 gives \( R = \frac{21}{5} = 4 \frac{1}{5} \)

This could have been done by changing the subject of the formula and then substituting values in, though the steps are almost identical. (see next example).

(c) Make \( R \) the subject of the formula \( V = \frac{2R}{R - r} \)

Multiply both sides by \( (R - r) \) [inverse of \( \div (R - r) \)]

Expand the bracket on the LHS \( V(R - r) = 2R \)

Add \( Vr \) to both sides [inverse of \( -Vr \)] \( VR - Vr = 2R \)

Subtract \( 2R \) from both sides [inverse of \( 2R \)] \( VR - 2R = Vr \)

Factorise the LHS \( R(V - 2) = Vr \)

Divide by \( (V - 2) \) [inverse of \( \times (V - 2) \)] \( R = \frac{Vr}{V - 2} \)

Note substituting the values from example (b) gives

\[
R = \frac{Vr}{V - 2} = \frac{7 \times 3}{7 - 2} = \frac{21}{5} = 4 \frac{1}{5} \text{ as above}
\]

(d) Make \( h \) the subject of the formula \( d = \sqrt{2hr} \)

It is first necessary to deal with the \( \sqrt{\ } \) and as the inverse of \( \sqrt{\ } \) is ‘square’

square both sides \( d^2 = 2hr \)

Divide both sides by \( 2r \) \( \frac{d^2}{2r} = h \)

This is the same as \( h = \frac{d^2}{2r} \)
(e) Make \( t \) the subject of the formula \( s = \frac{1}{2}ut^2 \)

multiply both sides by 2 \( 2s = ut^2 \)
divide both sides by \( u \) \( \frac{2s}{u} = t^2 \) or \( t^2 = \frac{2s}{u} \)

take the square root \([\text{inverse of ‘square’}]\) \( t = \pm \sqrt[2]{\frac{2s}{u}} \)

Notice the \( \pm \) sign, it is important! \( x^2 = 4 \Rightarrow x = \pm 2 \) but \( x = \sqrt{4} \Rightarrow x = 2 \)
The root sign means ‘take the positive value’ of the square root.

Exercise 2

1. Make \( x \) the subject in each of the following

\[
\begin{align*}
\text{a) } & \quad x + 2n - m = 0 \\
\text{b) } & \quad mx + nx = p \\
\text{c) } & \quad \frac{m}{x} = n \\
\text{d) } & \quad \frac{x}{m} + n = p \\
\text{e) } & \quad \frac{x}{m} = \frac{n}{p} \\
\text{f) } & \quad \frac{x}{r} = \frac{s}{t} + 1 \\
\text{g) } & \quad \sqrt{x} = m + n \\
\text{h) } & \quad p\sqrt{x} - 1 = r \\
\text{i) } & \quad \frac{m}{x} = \frac{n}{x-1} \\
\text{j) } & \quad \frac{m}{n-x} = \frac{n}{m-x} \\
\text{k) } & \quad \frac{x+m}{m} - \frac{x-n}{n} = \frac{x}{n} \\
\text{l) } & \quad \sqrt{mx+n} = m + \sqrt{n}
\end{align*}
\]

2. Given the formula \( b = 2a - T \)

(a) find the value of \( T \) when \( a = 7, b = 3 \). (b) make \( T \) the subject of the formula.

3. Given the formula \( 2a^2 = b^2 + B \)

(a) find the value of \( B \) when \( a = 3, b = -3 \) (b) make \( B \) the subject of the formula.

4. Given the formula \( D = \frac{d}{3+Q} \)

(a) find the value of \( Q \) when \( d = 1.3, D = 2.6 \) (b) make \( Q \) the subject of the formula.

5. The volume \( V \) (in \( \text{cm}^3 \)) of a cone with height \( h \) cm and base radius \( r \) cm is given by the formula \( V = \frac{1}{3} \pi r^2 h \).

(a) Calculate the radius of the base of cone with height 5cm and volume \( 33 \frac{1}{2} \text{cm}^3 \), taking \( \pi \) as 3.142. Give your answer correct to 2 decimal places.

(b) make \( r \) the subject of the formula

6. Make \( n \) the subject of the formula \( C = \frac{an + 1}{n} \)

7. Make \( t \) the subject in the formula \( v = u + at \).

8. Make \( h \) the subject of the formula \( S = ar(r + h) \).
9. Make $b$ the subject of the formula $a = \sqrt{\frac{b}{b+c}}$.

10. The formula $v = at + 16t^2$ is used in mechanics. Rewrite the formula to make $a$ the subject and find its value when $v = 300$, $t = 4$.

11. Given the formula $y = \frac{p+x}{1-px}$
   
   (a) find the value of $x$ when $y = 3$, $p = -2.4$  (b) make $x$ the subject of the formula.

   \[
   \begin{align*}
   \text{Exercise 1} \\
   1. 4 & 2. 1 & 3. 2 & 4. \frac{3}{5} & 5. -4 & 6. \frac{7}{2} \\
   13. 5 \frac{1}{4} & 14. 1 & 15. -\frac{2}{3} & 16. \frac{18}{5} = 3 \frac{3}{5} & 17. 0 & 18. \frac{2}{3} \\
   19. -2 & 20. 4 \frac{1}{2} & & & & \\
   \end{align*}
   \]

\text{Exercise 2}
1. \( a \) \( m - 2n \) \( b \) \( \frac{p}{m + n} \) \( c \) \( \frac{m}{n} \) \( d \) \( m(p - n) \)
\( e \) \( \frac{mn}{p} \) \( f \) \( \frac{r(s + t)}{t} \) \( g \) \( (m + n)^2 \) \( h \) \( \left( \frac{r}{p} \right)^2 + 1 = \frac{r^2 + p^2}{p^2} \)
\( i \) \( \frac{m}{m - n} \) \( j \) \( \frac{n^2 - m^2}{n - m} = n + m \) \( k \) \( \frac{2mn}{2m - n} \) \( l \) \( m + 2\sqrt{n} \)

2. \( T = 11; \ T = 2a - b \)

3. \( B = 9; \ B = 2a^2 - b^2 \)

4. \( Q = -2 \frac{1}{2}; \ Q = \frac{d - 3D}{D} \)

5. \( r = 2.53; \ r = \frac{3V}{\sqrt{\pi h}} \)

6. \( n = \frac{1}{C - a} \)

7. \( t = \frac{v - u}{a} \)

8. \( h = \frac{S - ar^2}{ar} = \frac{S}{ar} - r \)

9. \( b = \frac{-a^2c}{a^2 - 1} = \frac{a^2c}{1 - a^2} \)

10. \( a = \frac{v - 16r^2}{t} = \frac{v}{t} - 16r \); \( 11 \)

11. \( (a) \) \( x = -0.871 \) (to 3 decimal pl); 
\( (b) \) \( \frac{y - p}{1 + py} \)

We would appreciate your comments on this guide, especially if you’ve found any errors, so that we can improve it for future use. Please contact the Maths Skills Adviser by email at skills@hull.ac.uk

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