Differential Equations I

Mathematics Worksheet

This is one of a series of worksheets designed to help you increase your confidence in handling Mathematics. This worksheet contains both theory and exercises which cover 1st Order Differential Equations. It deals with 4 types:

1. Exact
2. Variables Separable
3. Homogeneous
4. Linear

There are often different ways of doing things in Mathematics and the methods suggested in the worksheets may not be the ones you were taught. If you are successful and happy with the methods you use it may not be necessary for you to change them. If you have problems or need help in any part of the work then there are a number of ways you can get help.

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What Are Differential Equations?

Any equation containing terms of the form $\frac{d^n y}{dx^n}$ is a differential equation.

The following are all differential equations of varying order and degree:

$$x \frac{dy}{dx} - y^2 = 0, \quad xy \frac{d^2 y}{dx^2} - y^2 \sin x = 0, \quad \left( \frac{dy}{dx} \right)^2 = \frac{d^3 y}{dx^3}.$$ 

Sometimes $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, $\frac{d^3 y}{dx^3}$ are written as $y'$, $y''$, $y'''$ to save space!

Order: If the highest derivative appearing is $\frac{dy}{dx}$, then the equation is of order 1.

If the highest is $\frac{d^2 y}{dx^2}$, then it is of order 2 etc. Not all such equations can be solved in terms of the unknown, some have to be solved by numerical methods giving approximate results.

Type 1 Exact

If you are asked to solve the Differential Equation $\frac{dy}{dx} = \cos x + 3x^2$ you can integrate immediately giving $y = \sin x + x^3 + A$. In the same way the following second order Differential Equation can be solved directly by integration:

$$\frac{d^2 y}{dx^2} = 3e^{3x} + 4x - \sin x$$

integrating $\frac{dy}{dx} = e^{3x} + 2x^2 + \cos x + A$

integrate again $y = \frac{e^{3x}}{3} + \frac{2x^3}{3} + \sin x + Ax + B$

Note that integrating twice has introduced the 2 arbitrary constants $A$ and $B$.

If you have a DE of the form $\frac{d^n y}{dx^n} = f(x)$ then you can, in theory, integrate $n$ times to find the solution. This assumes that you can integrate at each step (remember that you cannot integrate every function algebraically). Your solution will contain $n$ arbitrary constants.

Type 2 Variables Separable

Equations of the form $f(x)\frac{dy}{dx} + g(y) = 0$ can be solved by ‘separating the variables’ which means all the $x$-terms along with $dx$ can be put on one side of the equation and all the $y$-terms along with $dy$ can be put on the other side.
Examples

1: Find the general solution of the equation \( \frac{dy}{dx} = \frac{xy}{x^2 + 1} \)

This gives \( \frac{dy}{y} = \frac{x}{x^2 + 1} \, dx \)

Integrating \( \int \frac{dy}{y} = \int \frac{x}{x^2 + 1} \, dx \)

\( \ln|y| = \frac{1}{2} \ln|x^2 + 1| + A \)

and simplify, putting \( \ln B = A \)

\( \ln|y| = \ln \sqrt{x^2 + 1} + \ln B = \ln(B\sqrt{x^2 + 1}) \)

hence \( y = B\sqrt{x^2 + 1} \) is the general solution

2: Solve the equation \( \frac{4x^2 + 1}{y + 1} = xy \frac{dy}{dx} \) given \( y(1) = 0 \) (ie \( y = 0 \) when \( x = 1 \))

This can be written as \( \frac{4x^2 + 1}{x} \, dx = y(y + 1) \, dy \)

Simplify and Integrate \( \int \left(\frac{4x + 1}{x}\right) \, dx = \int \left(y^2 + y\right) \, dy \)

\( 2x^2 + \ln|x| = \frac{y^3}{3} + \frac{y^2}{2} + A \)

When \( x = 1, \ y = 0 \)

\( 2 + \ln 1 = 0 + 0 + A \Rightarrow A = 2 \)

Solution \( 2x^2 + \ln|x| = \frac{y^3}{3} + \frac{y^2}{2} + 2 \)

Exercise 1

Solve the following

1. \( x = y \frac{dy}{dx} \)
2. \( \frac{dy}{dx} = \frac{x^2 + 1}{y^2 + y} \)
3. \( \frac{2xy}{x^2 + 1} = \frac{dy}{dx} \)
4. \( \frac{dy}{dx} = \frac{x^2 + x}{y^2 + y} \)
5. \( \cos^2 x \frac{dy}{dx} = \cos^2 y \)
6. \( e^{x+y} \frac{dy}{dx} = 1 \)
7. \( x \frac{dy}{dx} = \frac{y - 1}{y + 1} - y \)
8. \( (x + 1)y = x \frac{dy}{dx} \), where \( y(1) = 1 \)
9. \( \tan x \frac{dy}{dx} = \cot y \) where \( y(0) = \frac{\pi}{2} \)
10. \( (\sin x + \cos x) \frac{dy}{dx} = \cos x - \sin x \)
11. Find the equation of the curve which has gradient function \( \frac{dy}{dx} = \frac{y}{x^2 - 1} \)

and passes through the point (2, -1).
Type 3 Homogeneous Equations

A homogeneous equation of the first order is one that can be written in the form
\[ \frac{dy}{dx} = f\left(\frac{y}{x}\right). \]
To test whether a function can be written in that form substitute \( y = vx \) and the function should reduce to one containing only \( v \).

For instance \( \frac{dy}{dx} = \frac{x^3y - x^2y^2}{4xy^3 - x^3y} \) becomes
\[ \frac{dy}{dx} = \frac{(vx^4 - v^2x^4)}{(4v^3x^4 - vx^4)} = \frac{(v - v^2)x^4}{4v^3 - v} = \frac{v - v^2}{4v^3 - v}. \]

The substitution \( y = vx \) simplifies the function from two variables to one so it seems logical to try the same substitution for \( \frac{dy}{dx} \). This is another technique which may have come about by ‘trial and error’, but helps us in this case, to solve equations.

Differentiating \( y = vx \) (using product rule) gives
\[ \frac{dy}{dx} = \frac{d(vx)}{dx} = \frac{dv}{dx}x + v \frac{dx}{dx} = x \frac{dv}{dx} + v. \]

So by substituting \( y = vx \) the differential equation
\[ \frac{dy}{dx} = f\left(\frac{y}{x}\right) \text{ becomes } x \frac{dv}{dx} + v = f(v) \]
which can be solved by separating the variables.

**Example 1** Solve the differential equation \( \frac{dy}{dx} = \frac{4y}{x} \)

Putting \( y = vx \) this becomes
\[ x \frac{dv}{dx} + v = \frac{4vx}{x} = 4v \Rightarrow x \frac{dv}{dx} = 3v \]

Separating the variables gives
\[ \frac{dv}{v} = \frac{3dx}{x} \]
Integrate \( \ln|v| = 3\ln|x| + c = \ln|Ax^3| \) where \( c = \ln A \)

giving \( v = Ax^3 \Rightarrow \frac{y}{x} = Ax^3 \Rightarrow y = Ax^4 \)

Check \( y = Ax^4 \Rightarrow \frac{dy}{dx} = 4Ax^3 = \frac{4Ax^4}{x} = \frac{4y}{x} \) which is where we started!
**Example 2**: Solve the equation \( \frac{dy}{dx} = \frac{y(x+y)}{x(y-x)} \) given \( y(1) = 1 \) (ie \( y = 1 \) when \( x = 1 \)).

Substitute \( y = vx \)

\[
x \frac{dv}{dx} + v = \frac{v(x + vx)}{x(vx - x)} = \frac{v(1 + v)x^2}{(v - 1)x^2}
\]

\[
x \frac{dv}{dx} = \frac{v(1 + v)}{(v - 1)} - v
\]

\[
x \frac{dv}{dx} = \frac{v(1 + v) - v(v - 1)}{v - 1} = \frac{2v}{v - 1}
\]

separate variables \( \left( \frac{v - 1}{v} \right) dv = \frac{2}{x} dx \)

integrate \( \int \left( 1 - \frac{1}{v} \right) dv = \int \frac{2}{x} dx = 2 \int \frac{dx}{x} \)

\[ v - \ln|v| = 2 \ln|x| + c \]

put \( c = \ln A \)

\[ v = 2 \ln|x| + \ln|v| + \ln A = \ln(Ax^2 v) \]

But \( y = vx \), hence \( \frac{y}{x} = \ln \left( Ax^2 \frac{y}{x} \right) \)

\[ = \ln(Axy) \]

or \( y = x \ln(Axy) \)

But \( y(1) = 1 \); substituting in gives \( 1 = \ln A \Rightarrow A = e \)

The solution can be written as \( y = x \ln(exy) \) or \( y = x \ln(exy) = x(\ln e + \ln xy) = x(1 + \ln xy) \)

It is important to realise that there are often many different ways of writing the final solution. It depends on how it is to be used as to which is the most useful form.
**Example 3**: Solve the equation \( \frac{dy}{dx} - y = \sqrt{x^2 - y^2} \)

this can be written as \( \frac{dy}{dx} = \frac{\sqrt{x^2 - y^2}}{x} + y \)

substitute \( y = vx \) \( \frac{dv}{dx} + v = \frac{\sqrt{x^2 - v^2x^2}}{x} + vx \)

\( = \frac{x\sqrt{1-v^2} + vx}{x} \)

\( \frac{dv}{dx} + v = \sqrt{1-v^2} + v \)

simplify \( \frac{x\,dv}{dx} = \sqrt{1-v^2} \)

separate the variables \( \int \frac{dv}{\sqrt{1-v^2}} = \int \frac{dx}{x} \)

integrate \( \sin^{-1} v = \ln|x| + c \)

put \( v = \frac{y}{x} \) and \( c = \ln A \) \( \sin^{-1} \frac{y}{x} = \ln|x| + c = \ln|Ax| \)

\( y = x \sin(\ln|Ax|) \)

**Example 4**: Solve the equation \( \frac{dy}{dx} = x + y \)

Rearrange equation \( \frac{dy}{dx} = \frac{x + y}{x} \)

putting \( y = vx \) gives \( \frac{x\,dv}{dx} + v = \frac{x + vx}{x} = 1 + v \)

simplify \( \frac{x\,dv}{dx} = 1 \)

separate the variables \( dv = \frac{dx}{x} \)

integrate \( v = \int \frac{1}{x} \,dx = \ln|x| + A \)

& put \( A = \ln B \) \( v = \ln|x| + \ln B = \ln|Bx| \)

put \( v = \frac{y}{x} \) \( \frac{y}{x} = \ln|Bx| \)

giving answer \( y = x \ln|Bx| \)
Exercise 2
Solve the following

1. \( \frac{dy}{dx} = \frac{x - 2y}{x} \)
2. \( \frac{dy}{dx} = \frac{x^2 + y^2}{xy} \)
3. \( 2x^2 \frac{dy}{dx} = x^2 + y^2 \)
4. \( xy^2 \frac{dy}{dx} = x^3 - y^3 \)
5. \( x \frac{dy}{dx} = y + \sqrt{x^2 - y^2} \)
6. \( y \frac{dy}{dx} = -x + 2y \)
7. \( x(x^2 + y^2) \frac{dy}{dx} = y(y^2 - x^2) \)
8. \( (x^3 + xy^2) \frac{dy}{dx} = (x^2y - y^3) \)

Type 4 - Linear Equations

A linear equation is one of the form \( \frac{dy}{dx} + P(x)y = Q(x) \). The method used to solve such equations depends, again, on noticing a particular pattern.

If \( z = ye^{\int f(x)} \) then differentiating (using the product rule and the chain rule)

\[
\frac{dz}{dx} = \frac{dy}{dx} e^{\int f(x)} + ye^{\int f(x)} f'(x) = e^{\int f(x)} \left( \frac{dy}{dx} + f'(x)y \right)
\]

When compared to the linear equation above, this suggests that if we can find an \( \int f(x) \) to multiply both sides of the linear equation by then we'll be able to integrate the left hand side immediately.

Multiplying the linear equation \( \frac{dy}{dx} + P(x)y = Q(x) \) by \( e^{\int f(x)} \) gives

\[
e^{\int f(x)} \left( \frac{dy}{dx} + P(x)y \right) = e^{\int f(x)} Q(x)
\]

Comparing left hand side with \( e^{\int f(x)} \left( \frac{dy}{dx} + f'(x)y \right) \) (which can be integrated immediately) shows that we want to find \( f(x) \) where

\[
f'(x) = P(x) \Rightarrow f(x) = \int P(x)dx
\]

i.e. multiply the equation by \( e^{\int f(x)} = e^{\int P(x)dx} \) which is called the Integrating Factor.

Example 1 Solve the equation \( \frac{dy}{dx} - 3y = e^{2x} \)

From above the Integrating Factor is \( e^{\int -3dx} = e^{-3x} \)

Multiplying through gives

\[
e^{-3x} \frac{dy}{dx} - 3e^{-3x}y = e^{-3x}e^{2x}
\]

The left hand side is the differential of \( ye^{-3x} \)

Check, using product rule,

\[
\frac{d(ye^{-3x})}{dx} = ye^{-3x} + y(-3e^{-3x}) = e^{-3x} \frac{dy}{dx} - 3e^{-3x} y
\]
From the original equation
\[
\frac{dy}{dx} - 3y = e^{2x}
\]

multiply through by \( e^{-3x} \)
\[
e^{-3x} \frac{dy}{dx} - 3e^{-3x}y = e^{-3x} e^{2x}
\]
this can be written as
\[
\frac{d(ye^{-3x})}{dx} = e^{-x}
\]
integrating
\[
ye^{-3x} = \int e^{-x} \, dx
\]
\[
= -e^{-x} + A
\]
or, multiplying through by \( e^{3x} \)
\[
y = e^{2x} + Ae^{3x}
\]

**Example 2** Given \( 2 \cos x \frac{dy}{dx} + 4 \sin x = \sin 2x \) find \( y(x) \) (ie \( y \) as a function of \( x \)), given that \( y(0) = \frac{\pi}{3} \) (ie \( y = 0 \) when \( x = \frac{\pi}{3} \)).

Divide through by \( 2 \cos x \) to write it in the form \( \frac{dy}{dx} + P(x)y = Q(x) \)
\[
\frac{dy}{dx} + \frac{2 \sin x}{\cos x} y = \frac{\sin 2x}{2 \cos x} = \frac{2 \sin x \cos x}{2 \cos x}
\]
\[
\frac{dy}{dx} + (2 \tan x)y = \sin x
\]
The equation is linear, integrating factor \( e^{\int 2 \tan x \, dx} \)
\[
\int 2 \tan x \, dx = 2 \ln |\sec x| = \ln \sec^2 x
\]
hence integrating factor is \( e^{\int 2 \tan x \, dx} = e^{\ln (\sec^2 x)} = \sec^2 x \)
multiply by I. F. \( \sec^2 x \frac{dy}{dx} + \left( 2 \sec^2 x \tan x \right)y = \sec^2 x \sin x \)
\[
\frac{d(y \sec^2 x)}{dx} = \sec^2 x \sin x = \frac{\sin x}{\cos^2 x}
\]
Integrate
\[
y \sec^2 x = \int \frac{\sin x}{\cos^2 x} \, dx = \frac{1}{\cos x} + A
\]
Multiply by \( \cos^2 x \)
\[
y = \cos x + A \cos^2 x
\]
But \( y = 0 \) when \( x = \frac{\pi}{3} \) hence \( 0 = \cos \frac{\pi}{3} + A \cos^2 \frac{\pi}{3} \)
\[
0 = \frac{1}{2} + \frac{1}{4} A \Rightarrow A = -2
\]
Solution \( y(x) = \cos x - 2 \cos^2 x \)
Example 3

Solve the equation \( \frac{dy}{dx} - y = x \)

The I. F. is \( e^{\int -1 \, dx} = e^{-x} \) giving \( e^{-x} \frac{dy}{dx} - ye^{-x} = xe^{-x} \)

hence \( \frac{d(ye^{-x})}{dx} = xe^{-x} \)

giving \( ye^{-x} = \int xe^{-x} \, dx \)

Integrating the right side by parts

\[ = -xe^{-x} - \int e^{-x} \, dx \]

\[ = -xe^{-x} - e^{-x} + A \]

giving \( y = Ae^x - x - 1 \)

Exercise 3

Solve the following

1. \( \frac{dy}{dx} + y = e^{-x} \)  
2. \( \frac{dy}{dx} - xy = 0 \)  
3. \( \frac{dy}{dx} + \frac{y}{x} = \frac{1}{x} \)

4. \( x \frac{dy}{dx} - y = x^2 \)  
5. \( \frac{dy}{dx} + y = x \)  
6. \( \frac{dy}{dx} - 2y = x \)

7. \( \frac{dy}{dx} + y \cot x = \cos x \)  
8. \( x \frac{dy}{dx} - 2y = x^3 \)

9. \( \frac{dy}{dx} - y \tan x = \cos^2 x \)  
10. \( (x^2 - 1)\frac{dy}{dx} + 2xy = x \)

11. \( (1 - x^2)\frac{dy}{dx} - xy = 1 \)  
12. \( \frac{dy}{dx} + 3y = e^{2x} \)

Exercise 4

Miscellaneous Differential Equations

Solve the following

1. \( \tan t \frac{dx}{dt} = x \)  
2. \( x \frac{dy}{dx} - y = x^2 \ln x \)  
3. \( \cos \theta = \sin \theta \frac{d\theta}{dr} \)

4. \( 3x \frac{dy}{dx} = x^2 + 2 \)  
5. \( 2x \frac{dy}{dx} + 8y = \frac{4}{x^2} \)  
6. \( 3 \frac{d^2 y}{dx^2} = 12x^2 + 2 \sin x \)

7. Find the solution of the differential equation \( \frac{dy}{dx} = xy \ln x \) which satisfies the conditions \( x = 1, \ y = 1 \). Give \( \ln y \) in terms of \( x \).

8. Solve the differential equation \( x \frac{dy}{dx} + 2y = e^x \) (\( x > 0 \)) given that \( y = 1 \) when \( x = 1 \).

9. Find the solution of the differential equation \( \frac{dy}{dx} + y \cot x = \sin x \) for which \( y = 1 \) when \( x = \pi \).
10. Find the solution of the differential equation \( ye^{2x} \frac{dy}{dx} = x \) which satisfies the boundary condition \( y = 2 \) when \( x = 0 \). Give \( y \) in terms of \( x \).

ANSWERS
(note many of the answers can be put in slightly different forms, depending whether the constant is put as \( A \) or \( \ln|B| \) or \( -\ln|C| \) etc.)

Exercise 1
1. \( 3x^2 = 2y^2 + A \)
2. \( x^4 + 2x^2 = 4y + A \)
3. \( \ln|x^2 + 1| = \ln y + A \) or \( y = A(x^2 + 1) \)
4. \( 2y^3 + 3y^2 = 2x^3 + 3x^2 + A \)
5. \( \tan y = \tan x + A \)
6. \( e^y + e^{-x} = A \)
7. \( \ln\left[x\sqrt{y^2 + 1}\right] + \tan^{-1} y = A \)
8. \( x + \ln x = 1 + \ln y \)
9. \( \sin x \cos y = 1 \)
10. \( y = \ln(\sin x + \cos x) + A \)
11. \( y^2(x + 1) = 3(x - 1) \)

Exercise 2
1. \( x^3 - 3x^2y = A \)
2. \( y^2 = 2x^2 \ln(Ax) \)
3. \( 2x = (y - x)\ln(Ax) \)
4. \( \ln x + \frac{1}{6}\ln\left[\frac{x^3 - 2y^3}{x^3}\right] = A \) or \( Bx^3\left(x^3 - 2y^3\right) = 1 \)
5. \( \sin^{-1} \frac{y}{x} = \ln x + A \) or \( \sin^{-1} \frac{y}{x} = \ln|Ax| \)
6. \( \ln\left[\frac{y-x}{x}\right] - \frac{x}{y-x} = -\ln|Ax| \) or \( \ln\left|\frac{y-x}{y-x}\right| = A \)
7. \( 2x^2 \ln|Ay| + y^2 = 0 \)
8. \( 2y^2 \ln|Ay| - x^2 = Cy^2 \)
Exercise 3

1. $y e^x = x + A$ or $y = (x + A)e^{-x}$
2. $y = Ae^{\frac{1}{2}x^2}$
3. $xy = x + A$ or $y = 1 + Ax^{-1}$
4. $y = x^2 + Ax$
5. $y = x - 1 + Ae^{-x}$
6. $4y + 2x + 1 = Ae^{2x}$
7. $y = \frac{1}{2} \sin x + \frac{A}{\sin x}$ or $4y \sin x + \cos 2x = A$
8. $y = x^3 + Ax^2$
9. $3y \cos x = 3 \sin x - \sin^3 x + A$
10. $2y(x^2 - 1) = x^2 + A$
11. $y \sqrt{1 - x^2} = \sin^{-1} x + A$
12. $5ye^{3x} = e^{5x} + A$

Exercise 4

1. $x = B \sin t$
2. $y = x^2 \ln|x| - x^2 + Ax$
3. $r = \ln|\sec \theta| + c$
4. $y = \frac{1}{6} x^2 + \frac{2}{3} \ln|Ax|$
5. $y = \frac{x^2 + A}{x^4}$
6. $y = \frac{x^4 - 2 \sin x}{3} + Ax + B$
7. $\ln y = \frac{1}{2} \left(x^2 \ln|x|\right) + \frac{1}{4} (1 - x^2)$
8. $y = \frac{xe^x - e^x + 1}{x^2}$
9. $y \sin x = \frac{1}{2} x - \frac{1}{4} \sin 2x - \frac{\pi}{2}$
10. $y^2 = \frac{1}{2} \left(9 - e^{-2x} - 2xe^{-2x}\right)$

We would appreciate your comments on this worksheet, especially if you’ve found any errors, so that we can improve it for future use. Please contact the Maths tutor by email at skills@hull.ac.uk

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